Faster integer multiplication using short lattice vectors

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Main result

$\mathrm{M}(n) :=$ time needed to multiply n-bit integers.

("time" $=$ # steps on a multi-tape Turing machine.)

Theorem (H., van der Hoeven, 2018)

There is an integer multiplication algorithm achieving

$$\mathrm{M}(n) = O(n \log n \, 4^{\log^* n}).$$

$\log^* x$ is the iterated logarithm:

$$\log^* x := \begin{cases} 0 & \text{if } x \leq 1, \\ 1 + \log^*(\log x) & \text{if } x > 1. \end{cases}$$

Example: $\log^*(e^{(e^{(e^{(e^{e})})})}) = 6.$

History of bounds for $\mathsf{M}(n)$

$<$ 1962	?	n^2
1962	Karatsuba	$n^{\log 3/\log 2} \approx n^{1.58}$
1963	Toom	$n\,2^{5\sqrt{\log_2 n}}$
1966	Schönhage	$n\,2^{\sqrt{2\log_2 n}}(\log n)^{3/2}$
1969	Knuth	$n\,2^{\sqrt{2\log_2 n}}\log n$
1971	Schönhage–Strassen	$n \log n \log\log n$
2007	Fürer	$n \log n\, K^{\log^* n}$ for some $K > 1$

Fürer's algorithm recurses from size n to size n' exponentially smaller than n. Number of recursion levels is $\log^* n + O(1)$.

The const-  K measures the "expansion factor" at each level.

Progress on the value of K

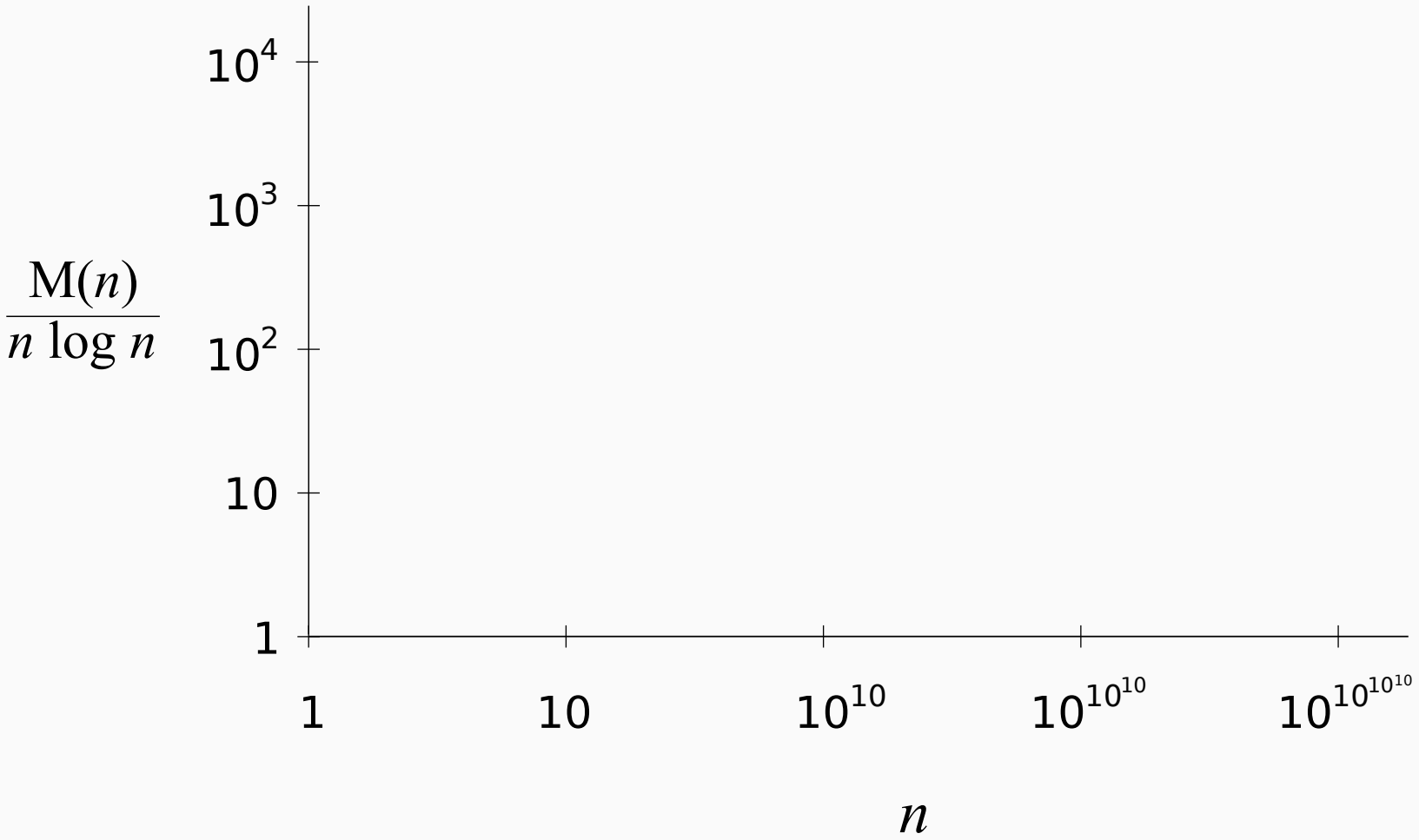
2014	H.–van der Hoeven–Lecerf	$K = 8$
†2014	H.–van der Hoeven–Lecerf	$K = 4$
†2015	Covanov–Thomé	$K = 4$
†2016	H.–van der Hoeven	$K = 4$
*2017	H.	$K = 6$
**2017	H.–van der Hoeven	$K = 4\sqrt{2} \approx 5.7$
2018	H.–van der Hoeven	$K = 4$

†: depends on unproved number-theoretic conjecture

*: submitted, under review

**: preprint, do not  -icipate publication

Implementation and performance



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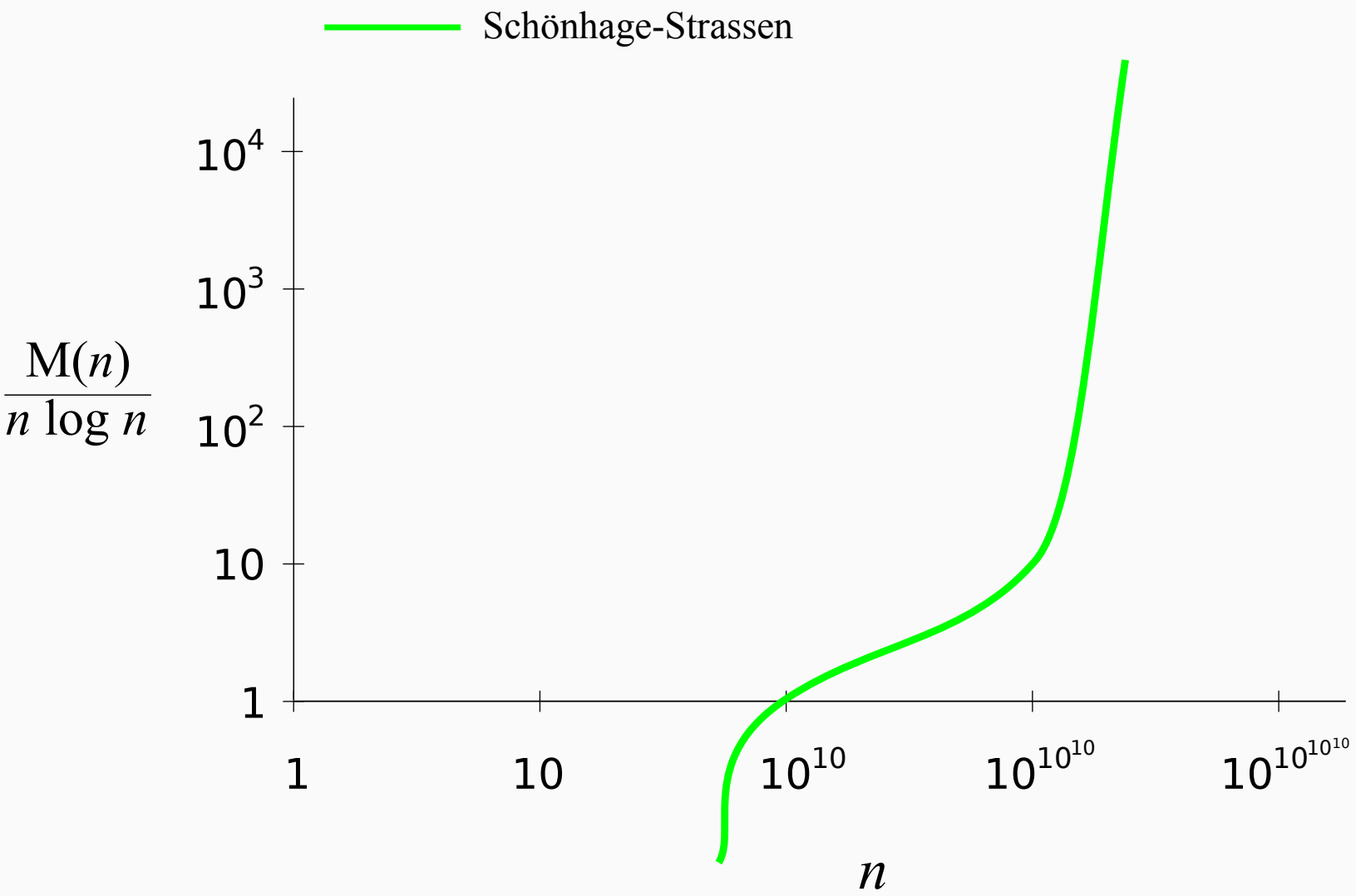

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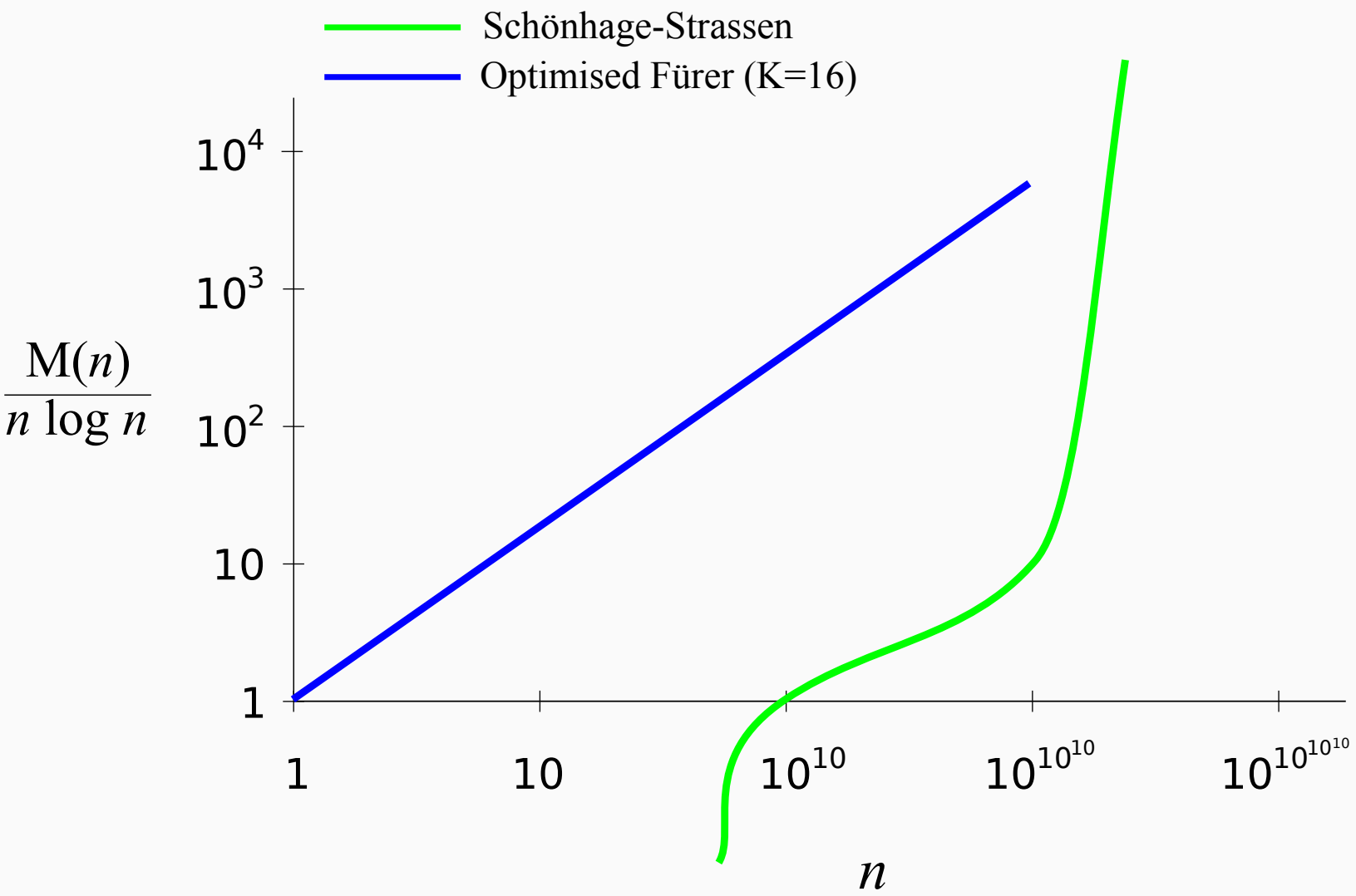

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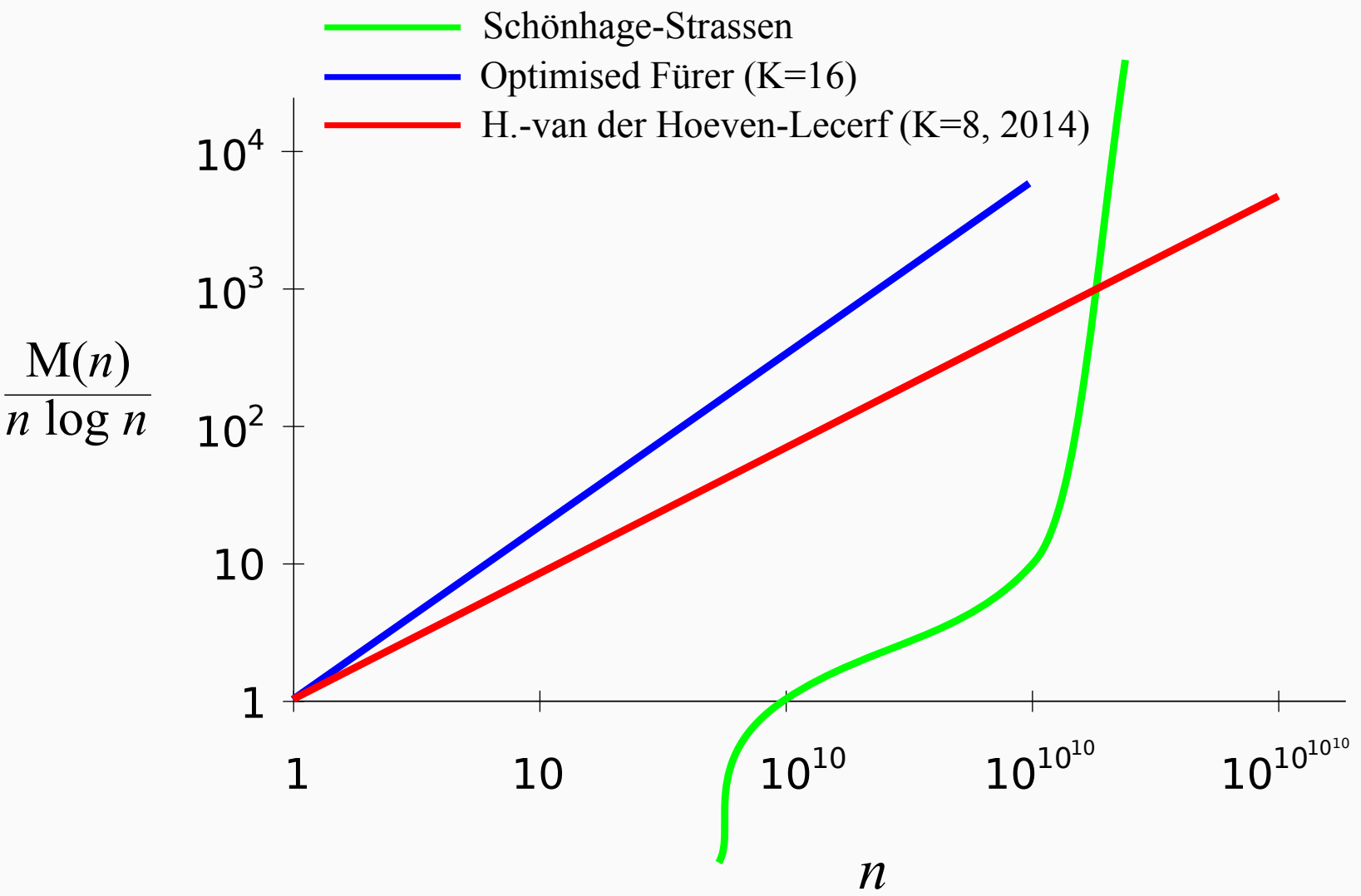

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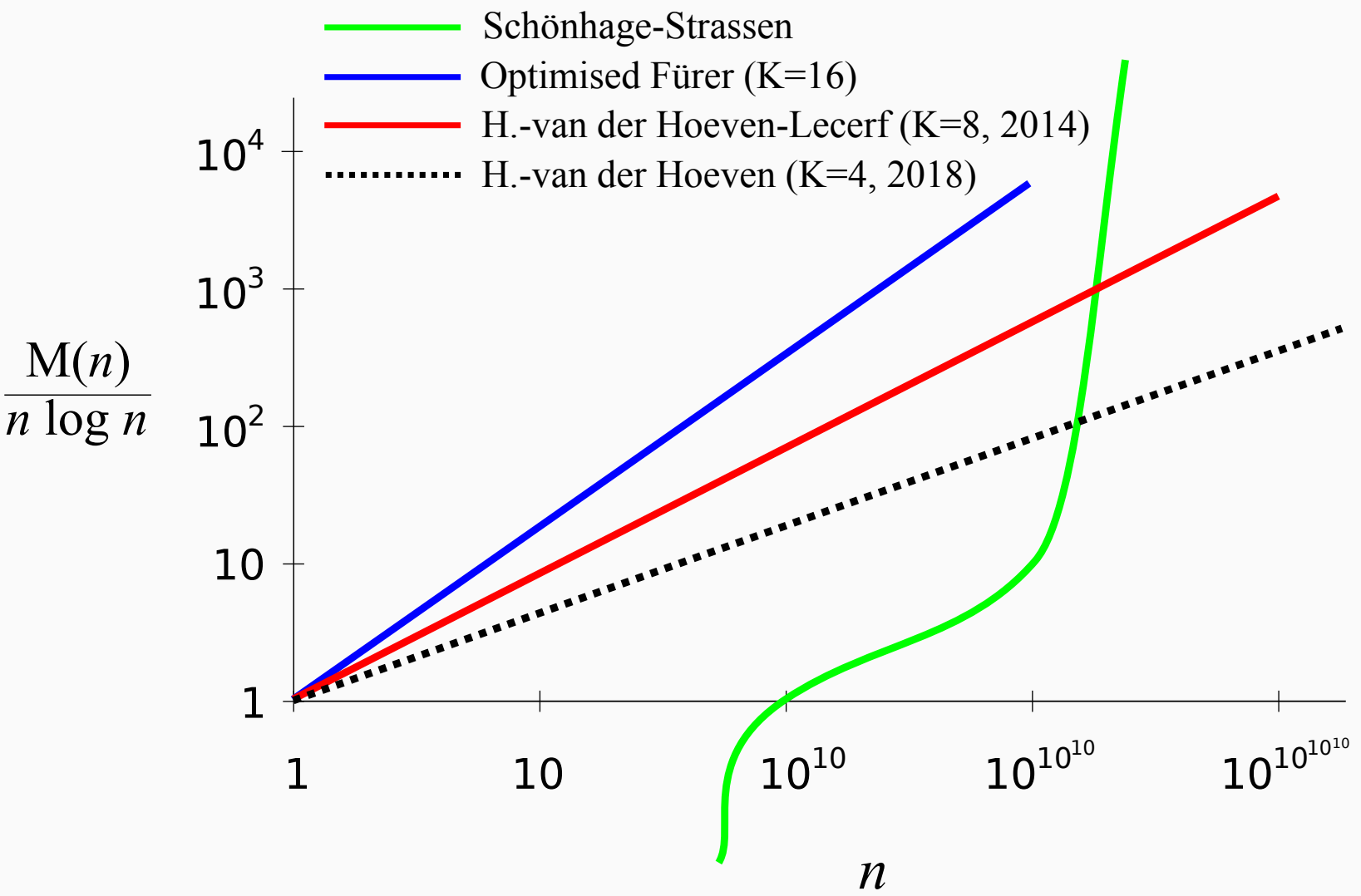

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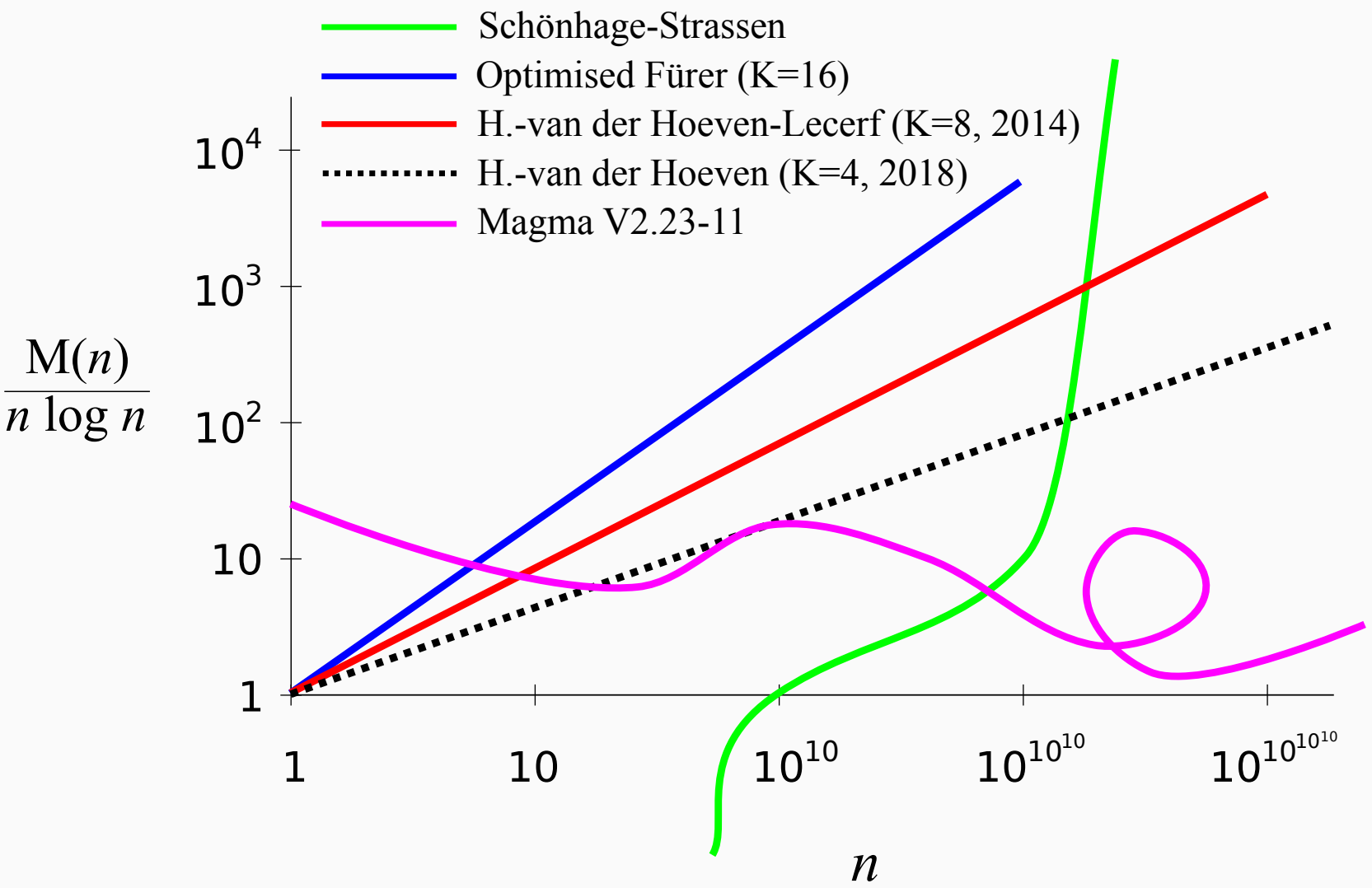

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4) Use Bluestein to reduce to "short" multiplications, size exponentially smaller than n.
5) Recurse!

Steps 1–3 are same as Fürer. Step 4 eliminates need for Fürer's "fast roots of unity".

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All of the $K = 4$ algorithms (both conditional and unconditional) attack (c) by replacing $\mathbf{C}$ with $\mathbf{Z}/q\mathbf{Z}$ for a suitable integer q.

Primes with cyclic structure

This may be adv-  -ageous if q has *cyclic structure* that maps efficiently onto FFTs.

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- H.–van der Hoeven (2016), plain vanilla FFT primes:

$$q = a \cdot 2^m + 1 \quad \Longrightarrow \quad x^m + a.$$

The new algorithm

The idea of the new algorithm is to manufacture cyclic structure for an almost *arbitrary* prime q.

Goal: reduce arithmetic in $\mathbf{Z}/q\mathbf{Z}$ to arithmetic in $\mathbf{Z}[x]/(x^m + 1)$, assuming that $q = 1 \pmod{2m}$.

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In the rest of the talk I will demonstrate the method for

$$q = 31415926535897932384626833, \qquad m = 4.$$

The condition $q = 1 \pmod 8$ guar-  -ees existence of a primitive 8-th root of unity modulo q:

$$\theta = 254253343156690445092273 5 \bmod q.$$

θ-representations

A *θ-representation* for $u \in \mathbf{Z}/q\mathbf{Z}$ is an expression

$$u = a_{m-1}\theta^{m-1} + \cdots + a_1\theta + a_0,$$

where the a_i are integers such that $|a_i| \leq mq^{1/m}$.

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For inst-  :

$$\begin{aligned} u &= 271828182845904523536028 \pmod{q} \\ &= 3366162 \cdot \theta^3 + 951670 \cdot \theta^2 - 5013490 \cdot \theta - 3202352. \end{aligned}$$

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Notice that the total bitsize of $(a_0, \ldots, a_{m-1})$ is about the same as the bitsize of q.

Proposed ANTS by-laws

⋮

By-law #691: *any speaker whose presentation includes, in the expert opinion of the Steering Committee, a distastefully excessive number of low-resolution bitmaps, cartoons, or puns, pertaining to formic acid, colony social behaviour, or any species of the order Hymenoptera (but not including termites, which are members of the order Blattodea and are more closely related to cockroaches), shall be banned from ANTS for a period of not more than twenty-four months; or in the case that the subsequent ANTS is held in New Zealand, eighteen months.*

⋮

Arithmetic on θ-representations

How do we multiply elements of $\mathbf{Z}/q\mathbf{Z}$ in θ-representation?

Suppose we want to multiply

$$\begin{aligned} u &= 1414213562373095048801689 \pmod{q} \\ &= 3740635 \cdot \theta^3 + 3692532 \cdot \theta^2 - 3089740 \cdot \theta + 4285386 \end{aligned}$$

by

$$\begin{aligned} v &= 1732050807568877293527447 \pmod{q} \\ &= 4629959 \cdot \theta^3 - 4018180 \cdot \theta^2 - 2839272 \cdot \theta - 3075767. \end{aligned}$$

Arithmetic on θ-representations

First multiply as polynomials in θ, using the relation $\theta^m = -1$:

$$uv = 102668685436\,25 \cdot \theta^3 - 371231948042\,09 \cdot \theta^2 - 4729783170300 \cdot \theta + 2658245912907\,8.$$

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Problem: this is not a θ-representation.

The coefficients are too big.

We need a *reduction algorithm* to make the coefficients small again, without changing the value modulo q.

Arithmetic on θ-representations

Key idea: precompute a polynomial $P(x)$ giving a nontrivial representation of zero:

$$0 = P(\theta) = -394297 \cdot \theta^3 - 927319 \cdot \theta^2 + 1136523 \cdot \theta - 292956.$$

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Finding $P(x)$ is equivalent to the problem of finding a short nonzero vector in the lattice

$$\Lambda := \{(a_3, \ldots, a_0) \in \mathbf{Z}^4 : a_3\theta^3 + \cdots + a_0 = 0 \pmod{q}\}.$$

In general, Minkowski's theorem guarantees the existence of such a vector with $|a_i| \leq q^{1/m}$ for all i.

(I used LLL to compute the example above.)

Arithmetic on θ-representations

Now we want to subtract an appropriate multiple of

$$0 = P(\theta) = -394297 \cdot \theta^3 - 927319 \cdot \theta^2 + 1136523 \cdot \theta - 292956$$

from

$$uv = 10266868543625 \cdot \theta^3 - 37123194804209 \cdot \theta^2 - 4729783170300 \cdot \theta + 26582459129078.$$

to make the coefficients small.

For technical reasons, we use a Montgomery-style modular reduction algorithm.

Arithmetic on θ-representations

Precompute an auxiliary prime r (around the size of $q^{1/m}$) such that $P(\theta)$ is invertible modulo r, and let $J(\theta)$ be its inverse.

In our example:

$$\begin{aligned} r &= 42602761, \\ J(\theta) &= 17106162 \cdot \theta^3 + 6504907 \cdot \theta^2 + 30962874 \cdot \theta + 8514380, \end{aligned}$$

so that

$$\begin{aligned} J(\theta)P(\theta) &= -29688032222177 \cdot \theta^3 + 32133728922904 \cdot \theta^2 \\ &\qquad + 19033763340253 \cdot \theta - 3695193078095 \\ &= 0 \cdot \theta^3 + 0 \cdot \theta^2 + 0 \cdot \theta + 1 \pmod{r} \\ &= 1 \pmod{r}. \end{aligned}$$

Arithmetic on θ-representations

Now we can compute the "Montgomery quotient":

$$\begin{aligned} Q(\theta) &:= (uv)J(\theta) \pmod{r} \\ &= 3932274 \cdot \theta^3 - 14729381 \cdot \theta^2 + 20464841 \cdot \theta - 11934644, \end{aligned}$$

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$$\begin{aligned} uv - Q(\theta)P(\theta) = 424307736538\,43 \cdot \theta^3 - 773147238\,13102 \cdot \theta^2 \\ + 169907905\,39259 \cdot \theta + 331448628\,52478. \end{aligned}$$

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By construction, this last polynomial is divisible by r.

Dividing by r, we get

$$\frac{uv}{r} = 995963 \cdot \theta^3 - 1814782 \cdot \theta^2 + 398819 \cdot \theta + 777998 \pmod{q}.$$

Finally, we have found a θ-representation for uv/r.

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One also needs algorithms for addition/subtraction in θ-representation, and for conversions between standard and θ-representation. All straightforward with the tools already described.

Thank you!