Ranks, 2-Selmer groups, and Tamagawa numbers of elliptic curves with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$-torsion

Stephanie Chan [1] Jeroen Hanselman [2] Wanlin Li [3]

[1]University College London

[2]Ulm University

[3]University of Wisconsin, Madison

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Average rank

Given an elliptic curve over $\mathbb{Q}$ in its minimal short Weierstrass form:

$$E : y^2 = x^3 + a_4 x + a_6, \quad a_4, a_6 \in \mathbb{Z},$$

define its naive height as: $h(E) := \max\{4|a_4|^3, 27a_6^2\}$.

Elliptic curves (isomorphism classes) can be ordered according to their naive height h, and it is natural to consider the average rank:

$$\lim_{N\to\infty} \frac{\sum_{h(E)\leq N} r(E)}{\#\{E : h(E) \leq N\}}.$$

A conjecture and a known result

Minimalist conjecture (Quadratic twist family by Goldfeld, 1979)

Half of all elliptic curves have rank 0 and half have rank 1.

This would mean that the average rank should tend to $1/2$.

Theorem (Bhargava–Shankar, 2015)

The average rank of all elliptic curves over $\mathbb{Q}$ *is at most* 0.885.

Database

Balakrishnan, Ho, Kaplan, Spicer, Stein, and Weigandt (2016) created a database of all elliptic curves with $h(E) < 2.7 \cdot 10^{10}$, a total of 238,764,310 curves. The database contains the minimal model, the torsion subgroup, the conductor, the Tamagawa product, the rank, and the size of the 2-Selmer group for each curve.

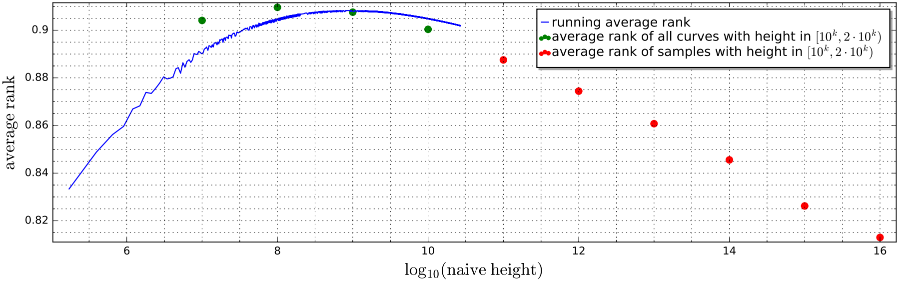

Figure: Average rank of elliptic curves over $\mathbb{Q}$ order by naive height.

Building a database for a special family

No elliptic curves recorded in the previous database have the largest possible torsion subgroup for $E/\mathbb{Q}$:

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}.$$

The first curve with such a torsion subgroup is

$$y^2 = x^3 - 1386747x + 368636886,$$

with $h(E) \approx 1.07 \cdot 10^{19}$.

We built a database for the family of elliptic curves with torsion subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$. We call this the $(2, 8)$-*torsion family*.

For each isomorphism class of the elliptic curves, we compute:
rank bounds, 2-Selmer rank, Tamagawa product and root number.

The special family

This family has a nice parametrisation:

$$E : y^2 = x(x+1)(x+u^4), \text{ where } u = \frac{2t}{t^2-1}, \; t \in \mathbb{Q} \cap (0,1).$$

Write the parameter $t = \frac{a}{b}$ for coprime integers a and b.

We work with *parameter height* $H(E) := \max\{|a|, |b|\}$, because

- it is easier to enumerate and compare curves;
- the curves have very large naive height:

$$0.559 \cdot h(E)^{1/48} < H(E) < 0.672 \cdot h(E)^{1/48}.$$

For each isomorphism class, we only consider the model with minimal H.

Why this family?

We chose the $(2, 8)$-torsion family because:

- we can quickly see curves of large naive height,
- its torsion structure makes it easier to carry out 2-power descent,
- it has interesting properties: e.g. it is not known whether this family has a curve of rank 4.

The $(2, 8)$-torsion family is 0% of all $E/\mathbb{Q}$. Average results on all elliptic curves over $\mathbb{Q}$ does not apply to the $(2, 8)$-torsion family, and vice versa.

Bounding rank — descent

The Selmer rank provides an upper bound on the rank, and is (in principle) computable.

Implementations used

- Sage: `mwrank` (Cremona's package) for 2-descent (with point search)
- Magma: `TwoPowerIsogenyDescentRankBound` (Fisher's method) for 2-power descent via isogeny (with point search)

Conditional upper bound — analytic rank

The Birch and Swinnerton-Dyer Conjecture (BSD)

The rank of an elliptic curve E equals to its analytic rank.

The *analytic rank* of E is the order of the zero of $L_E(s)$ at $s = 1$.

Computation with finite precision gives an upper bound on the rank.

Implementations used

- Sage: `analytic_rank_upper_bound` (Bober's method assumes GRH)
- Magma: `AnalyticRank`

If at any point the upper bound and lower bound differ by 1, assuming the Parity conjecture (special case of BSD) gives the actual rank of the curve.

Example

The last curve determined with $H < 100$ (at $t = 67/99$):

$$y^2 = x^3 - 190426278049230273017810263221931147x + 100937940155741627721354009888211066543347912564 1686$$

has naive height $\approx 10^{103}$.

- conductor $\approx 10^{20}$
- root number $= +1$
- Tamagawa product $= 67{,}108{,}864$

Example

The last curve determined with $H < 100$ (at $t = 67/99$):

$$y^2 = x^3 - 19042627804923027301781026322193147x \\ + 1009379401557416277213540098882110665433479125641686$$

has naive height $\approx 10^{103}$.

- conductor $\approx 10^{20}$
- root number $= +1$
- Tamagawa product $= 67{,}108{,}864$
- 2-Selmer rank $= 4$
- rank bounds:
  - `mwrank`: $0 \leq r \leq 2$;
  - `TwoPowerIsogenyDescentRankBound` (`MaxSteps`:= 6): $0 \leq r \leq 2$;

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  - `analytic_rank_upper_bound` ($\Delta := 4.0$): $r \leq 2$;
  - `AnalyticRank`: $r \leq 0$.

Average rank

The average rank seems to peak at 0.744 at $H = 24$, at the 121st curve.
All curves determined up to $H = 100$.
92.1% of curves determined for $H < 1000$.

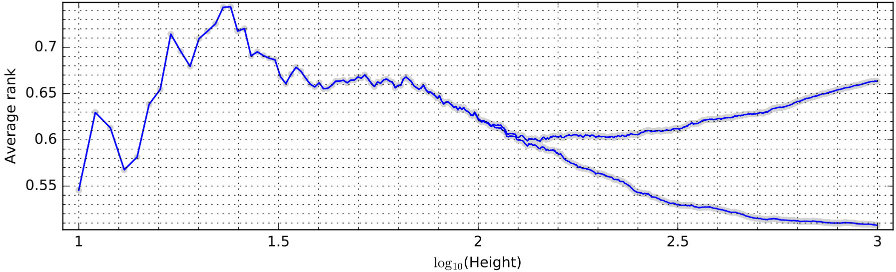


Figure: Average rank in the $(2, 8)$-torsion family.

Rank distribution

Rank	$H < 100$	$H < 250$	$H < 500$	$H < 1000$
0	43.3 %	45.0 %	43.8 %	41.8 %
1	51.1 %	49.5 %	49.7 %	50.1 %
2	5.6 %	2.4 %	0.9 %	0.3 %
3	0.2 %	0.1 %	0.0 %	0.0 %
≥ 4	0.0 %	0.0 %	0.0 %	0.0 %
Unknown	0.0 %	3.0 %	5.6 %	7.8 %
# curves	2000	12607	50565	202461
Av rank	0.626	$[0.545, 0.606]$	$[0.516, 0.628]$	$[0.508, 0.663]$

Table: Rank distribution up to different heights.

Average Tamagawa product

From the BHKSSW database:

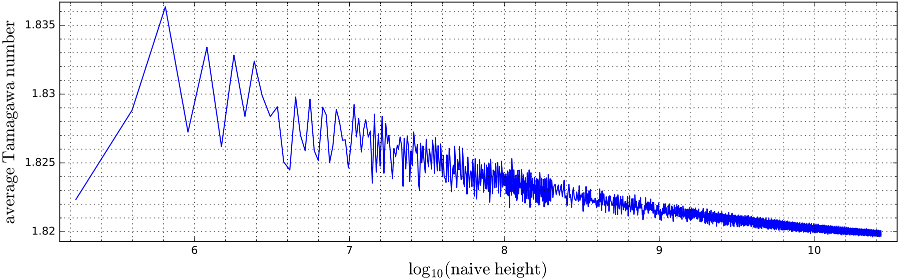
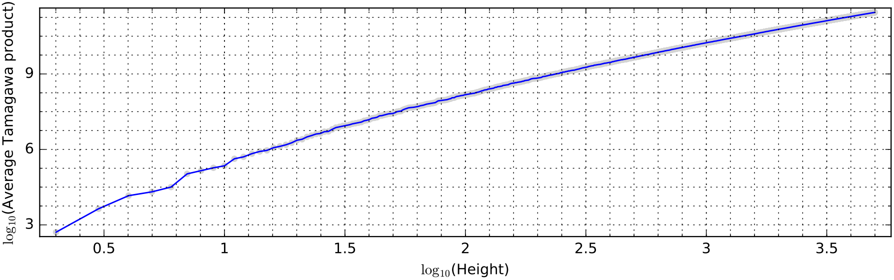


Figure: Average Tamagawa product of all elliptic curves order by height.



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Average Tamagawa product

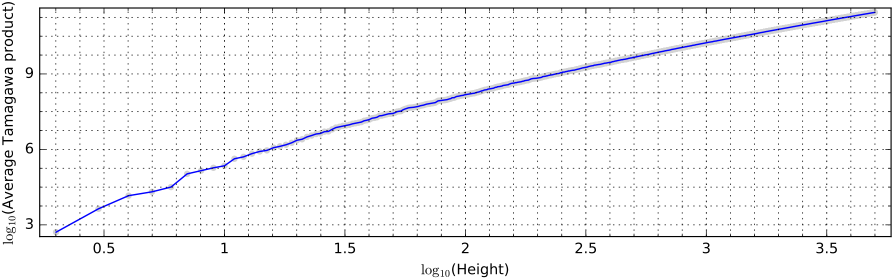

Figure: Average Tamagawa product in the $(2, 8)$-torsion family.

Theorem (C.–H.–L.)

The average Tamagawa product in the $(2, 8)$-torsion family up to height N has order of magnitude $(\log N)^{33}$.

Theorem (Bhargava–Shankar, 2013)

For $n \leq 5$, the average size of $\mathrm{Sel}_n(E)$ for all elliptic curves $E/\mathbb{Q}$ equals the sum of divisors of n.

For $n = 2$, from the BHKSSW database:

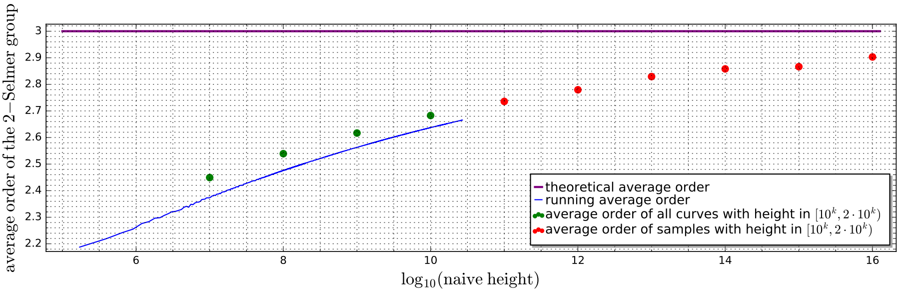

Figure: Average size of the 2-Selmer group of all elliptic curves order by height.

Average size of the 2-Selmer group

What about the $(2, 8)$-torsion family?
Is the average size of the 2-Selmer group converging?

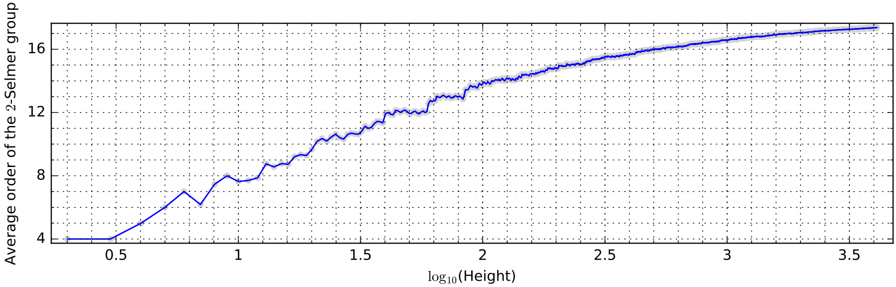

Figure: Average size of the 2-Selmer group in the $(2, 8)$-torsion family.

Maybe? Maybe not?

Average size of the 2-Selmer group

Theorem (Klagsbrun–Lemke Oliver, 2014)

The average size of $\mathrm{Sel}_2(E)$ *tends to infinity for the family of elliptic curves over* $\mathbb{Q}$ *with a 2-torsion point.*

Modifying their method, we proved:

Theorem (C.–H.–L.)

The average size of $\mathrm{Sel}_2(E)$ *tends to infinity in the* $(2, 8)$*-torsion family.*

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