# Effective aspects of quadratic Chabauty

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# Motivation

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That is, given a curve $X$ defined over $\mathbf{Q}$, how do we compute $X(\mathbf{Q})$? Can we make this algorithmic?

# Challenges in studying rational points on curves

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**Motivating problem** (Explicit Faltings): Given a curve $X/\mathbf{Q}$ with $g \geqslant 2$, compute $X(\mathbf{Q})$.

# Example 1: Can we compute $X(\mathbf{Q})$?

Consider $X$ with affine equation

$$y^2 = 82342800x^6 - 470135160x^5 + 52485681x^4 + 2396040466x^3 + 567207969x^2 - 985905640x + 247747600.$$

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It has at least **642** rational points*, with $x$-coordinates:

0, -1, 1/3, 4, -4, -3/5, -5/3, 5, 6, 2/7, 7/4, 1/8, -9/5, 7/10, 5/11, 11/5, -5/12, 11/12, 5/12, 13/10, 14/9, -15/2, -3/16, 16/15, 11/18, -19/12, 19/5, -19/11, -18/19, 20/3, -20/21, 24/7, -7/24, -17/28, 15/32, 5/32, 33/8, -23/33, -35/12, -35/18, 12/35, -37/14, 38/11, 40/17, -17/40, 34/41, 5/41, 41/16, 43/9, -47/4, -47/54, -9/55, -55/4, 21/55, -11/57, -59/15, 59/9, 61/27, -61/37, 62/21, 63/2, 65/18, -1/67, -60/67, 71/44, 71/3, -73/41, 3/74, -58/81, -41/81, 29/83, 19/83, 36/83, 11/84, 65/84, -86/45, -84/89, 5/89, -91/27, 92/21, 99/37, 100/19, -40/101, -32/101, -104/45, -13/105, 50/111, -113/57, 115/98, -115/44, 116/15, 123/34, 124/63, 125/36, 131/5, -64/133, 135/133, 35/136, -139/88, -145/7, 101/147, 149/12, -149/80, 75/157, -161/102, 97/171, 173/132, -65/173, -189/83, 190/63, 196/103, -195/196, -193/198, 201/28, 210/101, 227/81, 131/240, -259/3, 265/24, 193/267, 19/270, -279/281, 283/33, -229/298, -310/309, 174/335, 31/337, 400/129, -198/401, 384/401, 409/20, -422/199, -424/33, 434/43, -415/446, 106/453, 465/316, -25/489, 490/157, 500/317, -501/317, -404/513, -491/516, 137/581, 597/139, -612/359, 617/335, -620/383, -232/623, 653/129, 663/4, 583/695, 707/353, -772/447, 835/597, -680/843, 853/48, 860/697, 515/869, -733/921, -1049/33, -263/1059, -1060/439, 1075/21, -1111/30, 329/1123, -193/1231, 1336/1033, 321/1340, 1077/1348, -1355/389, 1400/11, -1432/359, -1505/909, 1541/180, -1340/1639, -1651/731, -1705/1761, -1757/1788, -1456/1893, -235/1983, -1990/2103, -2125/84, -2343/635, -2355/779, 2631/1393, -2639/2631, 396/2657, 2691/1301, 2707/948, -164/2777, -2831/508, 2988/43, 3124/395, -3137/3145, -3374/303, 3505/1148, 3589/907, 3131/3655, 3679/384, 535/3698, 3725/1583, 3940/939, 1442/3981, 865/4023, 2601/4124, -2778/4135, 1096/4153, 4365/557, -4552/2061, -197/4620, 4857/1871, 1337/5116, 5245/2133, 1007/5534, 1616/5553, 5965/2646, 6085/1563, 6101/1858, -5266/6303, -4565/6429, 6535/1377, -6613/6636, 6354/6697, -6908/2715, -3335/7211, 7363/3644, -4271/7399, -2872/8193, 2483/8301, -8671/3096, -6975/8941, 9107/6924, -9343/1951, -9589/3212, 10400/373, -8829/10420, 10511/2205, 1129/10836, 675/11932, 8045/12057, 12945/4627, -13680/8543, 14336/243, -100/14949, -15175/8919, 1745/15367, 16610/16683, 17287/16983, 2129/18279, -19138/1865, 19710/4649, -18799/20047, -20148/1141, -20873/9580, 21949/6896, 21985/6999, 235/25197, 16070/26739, 22991/28031, -33555/19603, -37091/14317, -2470/39207, 40645/6896, 46055/19518, -46925/11181, -9455/47584, 55904/8007, 39946/56827, -44323/57516, 15920/59083, 62569/39635, 73132/13509, 82315/67051, -82975/34943, 95393/22735, 14355/98437, 15121/102391, 130190/93793, -141665/55186, 39628/153245, 30145/169333, -140047/169734, 61203/171017, 148451/182305, 86648/195399, -199301/54169, 11795/225434, -84639/266663, 283567/143436, -291415/171792, -314333/195860, 289902/322289, 405523/327188, -342731/523857, 24960/630287, -665281/83977, -688283/82436, 199504/771597, 233305/795263, -799843/183558, -867313/1008993, 1142044/157607, 1399240/322953, -1418023/463891, 1584712/90191, 726821/2137953, 2224780/807321, -2849969/629081, -3198658/3291555, 675911/3302518, -5666740/2779443, 1526015/5872096, 13402625/4101272, 12027943/13799424, -71658936/86391295, 148596731/35675865, 58018579/158830656, 208346440/37486601, -1455780835/761431834, -3898675687/2462651894

Is this list complete?

*Noam Elkies and Michael Stoll (2008)

# Example 2: Can we compute $X(\mathbf{Q})$?

Consider $X$:

$$-x^3y+2x^2y^2-xy^3-x^3z+x^2yz+xy^2z-2xyz^2+2y^2z^2+xz^3-3yz^3=0.$$

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Question: Is this set of points above precisely $X(\mathbf{Q})$?

# Example 3: Can we compute $X(\mathbf{Q})$?

Consider $X$ with affine equation

$$y^2 = x(x-1)(x-2)(x-5)(x-6).$$

The *Chabauty–Coleman bound* tells us that

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Hence we have provably determined

$$X(\mathbf{Q}) = \{(0,0), (1,0), (2,0), (5,0), (6,0), (3,\pm 6), (10, \pm 120), \infty\}.$$

# Chabauty–Coleman

What is different in this last example? What allows us to compute $X(\mathbf{Q})$?

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What is different in this last example? What allows us to compute $X(\mathbf{Q})$?

(A bit of luck + ) satisfying an inequality between the **genus** of the curve $X$ and the **rank** of the Mordell-Weil group of its Jacobian $J(\mathbf{Q})$ (+ work of Chabauty and Coleman).

# Chabauty's theorem

**Theorem (Chabauty, '41)**

*Let $X$ be a curve of genus $g \geqslant 2$ over $\mathbf{Q}$. Suppose the Mordell-Weil rank $r$ of $J(\mathbf{Q})$ is less than $g$. Then $X(\mathbf{Q})$ is finite.*

To make Chabauty's theorem effective:

- Need to find a way to bound $X(\mathbf{Q}_p) \cap \overline{J(\mathbf{Q})}$
- Do this by constructing functions ($p$-adic integrals of 1-forms) on $J(\mathbf{Q}_p)$ that vanish on $\overline{J(\mathbf{Q})}$ and restrict them to $X(\mathbf{Q}_p)$

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This was done by Coleman (1985).

# The method of Chabauty–Coleman

Assume $X(\mathbf{Q}) \neq \emptyset$ and fix a basepoint $b \in X(\mathbf{Q})$.

- $\iota : X \hookrightarrow J$, sending $P \mapsto [(P) - (b)]$
- $p > 2$: prime of good reduction for $X$

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If $r < g$, there exists $\omega \in H^0(X_{\mathbf{Q}_p}, \Omega^1)$ such that

$$\int_b^P \omega = 0$$

for all $P \in X(\mathbf{Q})$. Thus by studying the zeros of $\int \omega$, we can find a finite set of $p$-adic points containing the rational points of $X$.

# Recap of the method (+bonus observations)

Given a curve $X/\mathbf{Q}$ of genus $g \geqslant 2$, embed it inside its *Jacobian J* and consider the rank $r$ of $J(\mathbf{Q})$.

- If $r < g$, we can use the Chabauty–Coleman method to compute a regular 1-form whose $p$-adic (Coleman) integral vanishes on rational points.
- By studying the zeros of this integral, Coleman gave the bound
$$\#X(\mathbf{Q}) \leqslant \#X(\mathbf{F}_p) + 2g - 2.$$
- This bound can be sharp in practice, as in the third example $(g = 2, r = 1, p = 7)$.
- Regardless, the Coleman integral cuts out a finite set of $p$-adic points; this set contains $X(\mathbf{Q})$ as a subset.
- Even when the bound is not sharp, we can often combine Chabauty–Coleman data at multiple primes (Mordell–Weil sieve) to extract $X(\mathbf{Q})$.

# Computing rational points via Chabauty–Coleman

We have

$$X(\mathbf{Q}) \subset X(\mathbf{Q}_p)_1 := \left\{ z \in X(\mathbf{Q}_p) : \int_b^z \omega = 0, \right\}$$

for a $p$-adic line integral $\int_b^* \omega$, with $\omega \in H^0(X_{\mathbf{Q}_p}, \Omega^1)$.

We would like to compute an annihilating differential $\omega$ and then calculate the finite set of $p$-adic points $X(\mathbf{Q}_p)_1$ .

# Example: Chabauty–Coleman with $g = 2, r = 1$

Suppose we have a genus 2 curve $X/\mathbf{Q}$ with $\operatorname{rk} J(\mathbf{Q}) = 1$ and $X(\mathbf{Q}) \neq \emptyset$. Fix a basepoint $b \in X(\mathbf{Q})$.

- We know $H^0(X_{\mathbf{Q}_p}, \Omega^1) = \langle \omega_0, \omega_1 \rangle$.
- Since $r = 1 < 2 = g$, we can compute $X(\mathbf{Q}_p)_1$ as the zero set of a $p$-adic integral.
- If we know one more point $P \in X(\mathbf{Q})$, we can compute the constants $A, B \in \mathbf{Q}_p$:

  $$\int_b^P \omega_0 = A, \quad \int_b^P \omega_1 = B,$$

  then solve the equation

  $$f(z) := \int_b^z (B\omega_0 - A\omega_1) = 0$$

  for $z \in X(\mathbf{Q}_p)$.
- The set of such $z$ is finite, and $X(\mathbf{Q})$ is contained in this set.

# Beyond Chabauty–Coleman

Do we have any hope of doing this when $r \geqslant g$?

- Conjecturally, yes, via Kim's nonabelian Chabauty program.
- Instead of using the Jacobian of $X$ and abelian integrals, use *nonabelian geometric objects* associated to $X$, which carry *iterated* Coleman integrals.
- These iterated integrals cut out Selmer varieties, which give a sequence of sets

  $$X(\mathbf{Q}) \subset \cdots \subset X(\mathbf{Q}_p)_n \subset X(\mathbf{Q}_p)_{n-1} \subset \cdots \subset X(\mathbf{Q}_p)_2 \subset X(\mathbf{Q}_p)_1$$

  where the depth $n$ set $X(\mathbf{Q}_p)_n$ is given by equations in terms of $n$-fold Coleman integrals

  $$\int_b^P \omega_n \cdots \omega_1.$$

- Note that $X(\mathbf{Q}_p)_1$ is the classical Chabauty–Coleman set.

# Nonabelian Chabauty

## Conjecture (Kim, ’12)

*For $n \gg 0$, the set $X(\mathbf{Q}_p)_n$ is finite.*

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**Questions:**

- When can $X(\mathbf{Q}_p)_n$ be shown to be finite?
- For which classes of curves can nonabelian Chabauty be used to prove Faltings’ theorem?

# Finiteness of $X(\mathbf{Q}_p)_n$

## Theorem (Coates–Kim '10)

*For $X/\mathbf{Q}$ with CM Jacobian, for $n \gg 0$, the set $X(\mathbf{Q}_p)_n$ is finite.*

## Theorem (Ellenberg–Hast '17)

*Can extend the above to give a new proof of Faltings' theorem for curves $X/\mathbf{Q}$ that are solvable Galois covers of $\mathbf{P}^1$.*

## Theorem (B.–Dogra '16)

*For $X/\mathbf{Q}$ with $g \geqslant 2$ and*

$$r < g + \operatorname{rk} NS(J_{\mathbf{Q}}) - 1,$$

*the set $X(\mathbf{Q}_p)_2$ is finite.*

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*the set $X(\mathbf{Q}_p)_2$ is finite.*

So when can we explicitly compute $X(\mathbf{Q}_p)_2$? We call this *quadratic Chabauty*.

# Quadratic Chabauty: $\mathbf{Q}$-points and $p$-adic heights

Basic strategy:

- Use "quadratic Chabauty" (Kim's nonabelian Chabauty at level 2) to compute $X(\mathbf{Q}_p)_2$, a finite set of $p$-adic points that contains all rational points on $X$.
- We know that $X(\mathbf{Q}_p)_2$ is finite when $r = g$ and $\operatorname{rk} NS(J) > 1$. The difficulty is in making this effective.
- The functions cutting out $p$-adic points can be expressed in terms of $p$-adic heights pairings; the key is to move from linear relations (as in Chabauty–Coleman) to bilinear relations.
- These $p$-adic heights have a natural interpretation in terms of $p$-adic differential equations, with relevant constants computed in terms of known rational points.

# From classical Chabauty to quadratic Chabauty

Recap: we can think of classical Chabauty as using linear relations among $\int_b^x \omega$ for $\omega \in H^0(X_{\mathbf{Q}_p}, \Omega^1)$, when $r < g$, i.e., understanding

$$X(\mathbf{Q}) \to X(\mathbf{Q}_p) \xrightarrow{AJ_b} H^0(X_{\mathbf{Q}_p}, \Omega^1)^*$$
$$x \mapsto (\omega \mapsto \int_b^x \omega).$$

The simplest generalization of Chabauty–Coleman comes from considering bilinear relations on $H^0(X_{\mathbf{Q}_p}, \Omega^1)^*$ when $r = g$. This motivates the notion of a *quadratic Chabauty function*.

# Quadratic Chabauty function

## Definition

A quadratic Chabauty function $\theta$ is a function $\theta : X(\mathbf{Q}_p) \to \mathbf{Q}_p$ such that:

1. On each residue disk, the map $(AJ_b, \theta) : X(\mathbf{Q}_p) \to H^0(X_{\mathbf{Q}_p}, \Omega^1)^* \times \mathbf{Q}_p$ is given by a power series.
2. There exist
   - an endomorphism $E$ of $H^0(X_{\mathbf{Q}_p}, \Omega^1)^*$,
   - a functional $c \in H^0(X_{\mathbf{Q}_p}, \Omega^1)^*$, and
   - a bilinear form

$$B : H^0(X_{\mathbf{Q}_p}, \Omega^1)^* \otimes H^0(X_{\mathbf{Q}_p}, \Omega^1)^* \to \mathbf{Q}_p$$

   such that for all $x \in X(\mathbf{Q})$,

$$\theta(x) - B(AJ_b(x), E(AJ_b(x)) + c) = 0.$$

# Quadratic Chabauty functions

## Lemma

*A quadratic Chabauty function induces a function* $F : X(\mathbf{Q}_p) \to \mathbf{Q}_p$ *such that* $F(X(\mathbf{Q})) = 0$ *and* $F$ *has finitely many zeros.*

# Quadratic Chabauty functions

## Lemma

*A quadratic Chabauty function induces a function $F : X(\mathbf{Q}_p) \to \mathbf{Q}_p$ such that $F(X(\mathbf{Q})) = 0$ and $F$ has finitely many zeros.*

- The goal is to make this explicit: need a quadratic Chabauty function: need an $E, c$, and need to solve for $B$.
- Solving for $B$ is very similar to solving for linear relations in Chabauty–Coleman.

# Quadratic Chabauty functions

**Lemma**

*A quadratic Chabauty function induces a function $F : X(\mathbf{Q}_p) \to \mathbf{Q}_p$ such that $F(X(\mathbf{Q})) = 0$ and $F$ has finitely many zeros.*

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We find quadratic Chabauty functions using $p$-adic height functions. As a warm-up, we'll use $p$-adic heights to find integral points on affine hyperelliptic curves when $r = g$.

# $p$-adic heights on Jacobians of curves (Coleman-Gross)

- Assume $X(\mathbf{Q}) \neq \emptyset$ and fix a basepoint $b \in X(\mathbf{Q})$.
- Let $\iota : X \hookrightarrow J$, sending $P \mapsto [P - b]$.

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The Coleman-Gross $p$-adic height

$$h : J(\mathbf{Q}) \to \mathbf{Q}_p$$

- is a quadratic form
- decomposes as a finite sum of local heights $h = \sum_v h_v$ over *finite* primes $v$
- $h_p$ is computed with respect to a splitting $s$ of the Hodge filtration on $H^1_{\mathrm{dR}}(X_{\mathbf{Q}_p})$
- work of Bernardi, Néron, Perrin-Riou, Schneider, Mazur-Tate, Coleman-Gross, Nekovář, Besser

*More generally, for $X$ defined over a number field $K$, also depends on a choice of idele class character

# Coleman-Gross global height as a sum of local heights

Using the Coleman-Gross $p$-adic height, we start to see

*global p-adic height as a sum of local heights*

playing a role in understanding rational points:

- The Coleman-Gross $p$-adic height pairing is a (symmetric) bilinear pairing

$$h : \mathrm{Div}^0(X) \times \mathrm{Div}^0(X) \to \mathbf{Q}_p,$$

  with $h = \sum_v h_v$
- We have $h(D, \mathrm{div}(g)) = 0$ for $g \in \mathbf{Q}(X)^\times$, so $h$ is well-defined on $J \times J$.
- Construction of local height $h_v$ depends on whether $v = p$ or $v \neq p$.
  - $v \neq p$: intersection theory
  - $v = p$: normalized differentials, Coleman integration

# Quadratic Chabauty (roughly)

Given a global $p$-adic height $h$, we study it on rational points:

$$\underbrace{h}_{\substack{\text{quadratic form, rewrite as a}\\ p\text{-adic analytic function}\\ \text{using Coleman integrals}}} = \underbrace{h_p}_{\substack{p\text{-adic analytic function}\\ \text{via } p\text{-adic differential equation}}} + \underbrace{\sum_{v \neq p} h_v}_{\substack{\text{takes on finite}\\ \text{number of values}\\ \text{on rational points}\\ \text{(we can sometimes prove)}}}$$

For example, using the Coleman-Gross $p$-adic height, the statement of quadratic Chabauty for integral points has, as its main ideas, (1) *computing the local height $h_p$ as a double Coleman integral* and (2) *controlling* the finite number of values

$$\sum_{v \neq p} h_v(z - b, z - b)$$

takes on integral points $z$.

# Quadratic Chabauty for integral points

We use these double and single Coleman integrals to rewrite the global $p$-adic height pairing $h$ and to study it on integral points:

$$\underbrace{h}_{\substack{\text{quadratic form, rewrite as a}\\ p\text{-adic analytic function}\\ \text{using Coleman integrals}}} = \underbrace{h_p}_{\substack{p\text{-adic analytic function}\\ \text{via double Coleman integral}}} + \underbrace{\sum_{v \neq p} h_v}_{\substack{\text{takes on finite}\\ \text{number of values}\\ \text{on integral points}}}$$

# Quadratic Chabauty for integral points

We use these double and single Coleman integrals to rewrite the global $p$-adic height pairing $h$ and to study it on integral points:

$$\underbrace{h_p}_{\text{$p$-adic analytic function via double Coleman integral}} - \underbrace{h}_{\text{quadratic form, rewrite as a $p$-adic analytic function using Coleman integrals}} = -\underbrace{\sum_{v \neq p} h_v}_{\text{takes on finite number of values on integral points}}$$

# Quadratic Chabauty for integral points

## Theorem (B.-Besser-Müller)

*Let* $X/\mathbf{Q}$ *be a hyperelliptic curve. If* $r = g \geqslant 1$ *and* $f_i(x) := \int_b^x \omega_i$ *for* $\omega_i \in H^0(X_{\mathbf{Q}_p}, \Omega^1)$ *are linearly independent, then there is an explicitly computable finite set* $S \subset \mathbf{Q}_p$ *and explicitly computable constants* $\alpha_{ij} \in \mathbf{Q}_p$ *such that*

$$\theta(P) - \sum_{0 \leqslant i \leqslant j \leqslant g-1} \alpha_{ij} f_i f_j(P),$$

*takes values in* $S$ *on integral points, where* $\theta(P) = \sum_{i=0}^{g-1} \int_b^P \omega_i \bar{\omega}_i$.

This gives a quadratic Chabauty function $\theta$ and a finite set of values $S$ (giving a *quadratic Chabauty pair*).

How can we use these ideas to study rational points?

# From integral to rational points

Main problem generalizing this to rational points: we can't control $h_v(x)$ for $v \neq p$ when $x$ is rational but not integral.

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Workaround for rational points:

- Construct a QCF by associating to points of $X$ certain $p$-adic Galois representations, and then take Nekovář $p$-adic heights .
- Idea is to construct a representation $A_Z(b, x)$ for every $x \in X(\mathbf{Q})$. Depends on a choice of "nice" correspondence $Z$ on $X$. Such a correspondence exists when $\operatorname{rk} NS(J) > 1$.
- Restrict to case of $X$ with everywhere potential good reduction, then for all $v \neq p$, local heights $h_v(A_Z(b, x))$ are trivial.
- Compute $p$-adic height of $A_Z(b, x)$ via explicit description of $D_{cris}(A_Z(b, x))$ as a filtered $\phi$-module.

# Quadratic Chabauty for rational points

- Using Nekovář's $p$-adic height $h$, there is a local decomposition

$$h(A_Z(b,x)) = h_p(A_Z(b,x)) + \sum_{v \neq p} h_v(A_Z(b,x))$$

  where
  1. $x \mapsto h_p(A_Z(b,x))$ extends to a locally analytic function $\theta : X(\mathbf{Q}_p) \to \mathbf{Q}_p$ by Nekovář's construction and
  2. For $v \neq p$ the local heights $h_v(A_Z(b,x))$ are trivial since by assumption, all primes $v \neq p$ are of potential good reduction

This gives a QCF whose pairing is $h$ and whose endomorphism is induced by $Z$.

# Quadratic Chabauty

Suppose $X/\mathbf{Q}$ satisfies

- $r = g$,
- $\operatorname{rk} NS(J_{\mathbf{Q}}) > 1$,
- $p$-adic closure $\overline{J(\mathbf{Q})}$ has finite index in $J(\mathbf{Q}_p)$,
- $X$ has everywhere potential good reduction,
- and that we know enough rational points $P_i \in X(\mathbf{Q})$.

If we can solve the following problems, we have an algorithm for computing a finite subset of $X(\mathbf{Q}_p)$ containing $X(\mathbf{Q})$:

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If we can solve the following problems, we have an algorithm for computing a finite subset of $X(\mathbf{Q}_p)$ containing $X(\mathbf{Q})$:

1. Expand the function $x \mapsto h_p(A_Z(b,x))$ into a $p$-adic power series on every residue disk.
2. Evaluate $h(A_Z(b,P_i))$ for the known rational points $P_i \in X(\mathbf{Q})$.

Note that since we are assuming we have everywhere potentially good reduction, we have

$$h(A_Z(b,x)) = h_p(A_Z(b,x)),$$

i.e., the second problem is subsumed by the first.

# Working example: a special curve

**Serre's uniformity problem:** Does there exist an absolute constant $p_0$ such that for any non-CM elliptic curve $E/\mathbf{Q}$ and any prime $p > p_0$, the Galois representation

$$\bar{\rho}_{E,p} : G_{\mathbf{Q}} \to \mathrm{Aut}(E[p]) \cong \mathbf{GL}_2(\mathbf{F}_p)$$

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Maximal subgroups of $\mathbf{GL}_2(\mathbf{F}_p)$:

(1) Borel subgroups

(2) Exceptional subgroups

(3) Normalizers of split Cartan subgroups

(4) Normalizers of non-split Cartan subgroups

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Known for (1), (2), (3).

# Serre's uniformity problem, rational points on curves

For every $p$, there is a modular curve $X_{\mathrm{s}}(p)$ (resp. $X_{\mathrm{ns}}(p)$) such that for an elliptic curve $E/\mathbf{Q}$:

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For every $p$, there is a modular curve $X_s(p)$ (resp. $X_{ns}(p)$) such that for an elliptic curve $E/\mathbf{Q}$:

If $\mathrm{im}(\bar{\rho}_{E,p})$ is contained in the normalizer of a split (resp. non-split) Cartan subgroup of $\mathbf{GL}_2(\mathbf{F}_p)$, we get a point $P \in X_s(p)(\mathbf{Q})$ (resp. $P \in X_{ns}(p)(\mathbf{Q})$) which is not a cusp.

- Bilu–Parent ('11): There exists $p_0$ such that $X_s(p)(\mathbf{Q}) = \{\text{cusps, CM-points}\}$ for $p > p_0$.
- Bilu–Parent–Rebolledo ('13): $X_s(p)(\mathbf{Q}) = \{\text{cusps, CM-points}\}$ for $p \geqslant 11$, $p \neq 13$.

# The curve

One curve remained after the work of Bilu–Parent–Rebolledo: the split Cartan case of $p = 13$. The curve $X_{\mathrm{s}}(13)$ is known to be

- non-hyperelliptic (smooth plane quartic) with $r \geqslant g = 3$,
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We would like to compute $X_{\mathrm{s}}(13)(\mathbf{Q})$. A model for $X$ is given by

$$x^3y+x^3z-2x^2y^2-x^2yz+xy^3-xy^2z+2xyz^2-xz^3-2y^2z^2+3yz^3 = 0.$$

# Aside: non-hyperelliptic difficulties

One computational difficulty: need action of Frobenius to solve $p$-adic differential equations and compute Coleman integrals.

# Aside: non-hyperelliptic difficulties

One computational difficulty: need action of Frobenius to solve $p$-adic differential equations and compute Coleman integrals.

- In the case of hyperelliptic curves, can do this by reworking Kedlaya's zeta function algorithm for hyperelliptic curves over finite fields (computed in terms of $p$-adic Monsky-Washnitzer cohomology).
- Tuitman ('14,'15) gave an algorithm vastly generalizing Kedlaya's zeta function algorithm from hyperelliptic curves to smooth curves.
  - Main ideas: pick a nice lift of Frobenius, work in rigid cohomology, do the bookkeeping in terms of a map $x : X \to \mathbf{P}^1$.

# An interesting curve

We are interested in the following smooth plane quartic:

$$X: x^3y + x^3z - 2x^2y^2 - x^2yz + xy^3 - xy^2z + 2xyz^2 - xz^3 - 2y^2z^2 + 3yz^3 = 0.$$

The Jacobian $J$ of $X$ has $r = \operatorname{rk} J(\mathbf{Q}) = 3$. The set $X(\mathbf{Q})$ contains the following 7 rational points (Galbraith):

$$(0:1:0), (0:0:1), (-1:0:1),$$

$$(1:0:0), (1:1:0), (0:3:2), (1:0:1).$$

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**Question:** Does $X(\mathbf{Q})$ have more rational points?

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**Question:** Does $X(\mathbf{Q})$ have more rational points?

Since $r = g = 3$ and $\operatorname{rk} NS(J_{\mathbf{Q}}) = 3$, we can carry out quadratic Chabauty on this curve.

# High-level strategy: our favorite curve

Practical matters:

- Make a small change of coordinates to work with the following curve $X$:

$$Q(x,y) = y^4 + 5x^4 - 6x^2y^2 + 6x^3z + 26x^2yz + 10xy^2z - 10y^3z - 32x^2z^2 - 40xyz^2 + 24y^2z^2 + 32xz^3 - 16yz^3 = 0$$

  so that we have enough (5 of the known) rational points in each of two affine patches (and satisfy hypotheses of Tuitman's algorithm).

- Since $\operatorname{rk} NS(J_{\mathbf{Q}}) = 3$, we have two independent nontrivial nice correspondences $Z_1, Z_2$ on $X$; we compute equations for 17-adic heights $h^{Z_1}, h^{Z_2}$ on $X$
- Check the simultaneous solutions of the above two equations...are they precisely on the 7 known rational points?!

# Recovered zeros: first affine patch

| $X(\mathbf{F}_{17})$ | $\det(T_1(x)) = 0$ | $\det(T_2(x)) = 0$ |
|---|---|---|
| $(0,0)$ | $\mathbf{0}$ | $\mathbf{0}$ |
| $(0,2)$ | $\mathbf{0}$ | $\mathbf{0}$ |
| | $9 \cdot 17 + 11 \cdot 17^2 + 11 \cdot 17^3$ | |
| $(4,1)$ | | $4 + 13 \cdot 17 + 11 \cdot 17^2 + 4 \cdot 17^3 + 12 \cdot 17^4$ |
| | | $4 + 15 \cdot 17 + 4 \cdot 17^2 + 2 \cdot 17^3 + 11 \cdot 17^4$ |
| $(5,-2)$ | $5 + 14 \cdot 17 + 7 \cdot 17^2 + 13 \cdot 17^3 + 12 \cdot 17^4$ | |
| | | $5 + 15 \cdot 17 + 10 \cdot 17^2 + 8 \cdot 17^3 + 8 \cdot 17^4$ |
| | | $5 + 3 \cdot 17 + 8 \cdot 17^2 + 4 \cdot 17^3 + 2 \cdot 17^4$ |
| $(7,-7)$ | $\mathbf{7 + 8 \cdot 17 + 8 \cdot 17^2 + 8 \cdot 17^3 + 8 \cdot 17^4}$ | $\mathbf{7 + 8 \cdot 17 + 8 \cdot 17^2 + 8 \cdot 17^3 + 8 \cdot 17^4}$ |
| | $7 + 6 \cdot 17 + 10 \cdot 17^2 + 14 \cdot 17^3 + 15 \cdot 17^4$ | |
| | | $7 + 7 \cdot 17 + 8 \cdot 17^3 + 7 \cdot 17^4$ |
| $(7,6)$ | $7 + 8 \cdot 17 + 13 \cdot 17^2 + 3 \cdot 17^3 + 3 \cdot 17^4$ | |
| $(8,0)$ | $8 + 8 \cdot 17 + 12 \cdot 17^2 + 16 \cdot 17^3 + 7 \cdot 17^4$ | |
| | $8 + 16 \cdot 17 + 16 \cdot 17^2 + 17^3 + 9 \cdot 17^4$ | |
| | | $8 + 10 \cdot 17 + 7 \cdot 17^2 + 2 \cdot 17^3 + 10 \cdot 17^4$ |
| | | $8 + 5 \cdot 17 + 16 \cdot 17^2 + 6 \cdot 17^3 + 12 \cdot 17^4$ |
| $(8,-14)$ | $8 + 8 \cdot 17 + 14 \cdot 17^2 + 8 \cdot 17^3 + 16 \cdot 17^4$ | |
| | $8 + 10 \cdot 17 + 12 \cdot 17^2 + 2 \cdot 17^3 + 8 \cdot 17^4$ | |
| $(9,-4)$ | $9 + 10 \cdot 17 + 8 \cdot 17^2 + 4 \cdot 17^3 + 6 \cdot 17^4$ | |
| $(9,-8)$ | $\mathbf{9 + 8 \cdot 17 + 8 \cdot 17^2 + 8 \cdot 17^3 + 8 \cdot 17^4}$ | $\mathbf{9 + 8 \cdot 17 + 8 \cdot 17^2 + 8 \cdot 17^3 + 8 \cdot 17^4}$ |
| | $9 + 6 \cdot 17^2 + 6 \cdot 17^3 + 3 \cdot 17^4$ | |
| $(13,-8)$ | $13 + 12 \cdot 17 + 7 \cdot 17^2 + 2 \cdot 17^3 + 12 \cdot 17^4$ | |
| | $13 + 13 \cdot 17 + 7 \cdot 17^2 + 7 \cdot 17^3 + 13 \cdot 17^4$ | |
| | | $13 + 4 \cdot 17 + 10 \cdot 17^2 + 3 \cdot 17^3 + 7 \cdot 17^4$ |
| $(15,-3)$ | $15 + 8 \cdot 17 + 9 \cdot 17^2 + 4 \cdot 17^3 + 16 \cdot 17^4$ | |
| | $15 + 4 \cdot 17 + 5 \cdot 17^2 + 2 \cdot 17^4$ | |

Upshot: recovered four points in $X_s(13)$, proved no other **Q**-points in the first affine patch. (A total of five **Q**-points here, as we'd hoped!)

# Rational points on $X_s(13)$

Repeating this for the second affine patch similarly shows that there are no new rational points found.

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By the work of Baran, we know $X_s(13)$ is isomorphic to $X_{ns}(13)$ over $\mathbf{Q}$, so we also get (for free) that $|X_{ns}(13)(\mathbf{Q})| = 7$.

# A quadratic Chabauty bound

Recall that Coleman showed, via a study of zeros of an integral defining $X(\mathbf{Q}_p)_1$, that when $r < g$, we have

$$\#X(\mathbf{Q}) \leqslant \#X(\mathbf{F}_p) + 2g - 2.$$

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Can similarly study zeros of a quadratic Chabauty function to prove

## Theorem (B.–Dogra)

*Suppose* $r = g$ *and* $\operatorname{rk} NS(J_{\mathbf{Q}}) > 1$. *Let* $\kappa_p = 1 + \frac{p-1}{p-2}\frac{1}{\log(p)}$. *Then*

$$\#X(\mathbf{Q}) < \kappa_p \left( \prod_{bad\ v} n_v \right) \#X(\mathbf{F}_p)(16g^3 + 15g^2 - 16g + 10).$$