Computing normalizers of tiled orders in $M_n(\mathbb{Q}_p)$

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Goal

We want to compute the normalizer of a tiled order $\Gamma \subset M_n(\mathbb{Q}_p)$*. We accomplish this by realizing the normalizer as the symmetries of a certain convex polytope, and later as the automorphisms of a valued quiver.*

Maximal orders in $M_n(\mathbb{Q}_p)$

Let $V = \mathbb{Q}_p^n$, fix a basis $\{e_1, e_2, \ldots, e_n\}$ and identify $B = M_n(\mathbb{Q}_p)$ with $\mathrm{End}_{\mathbb{Q}_p}(V)$.

1. Given $L_0 = \mathbb{Z}_p e_1 \oplus \mathbb{Z}_p e_2 \oplus \cdots \oplus \mathbb{Z}_p e_n$, we identify $\Lambda_0 := M_n(\mathbb{Z}_p)$ with $\mathrm{End}_{\mathbb{Z}_p}(L_0)$.

2. For any maximal order $\Lambda \subset B$, there exists $\xi \in B^\times$ such that $\Lambda = \xi \Lambda_0 \xi^{-1}$, and we identify Λ with $\mathrm{End}_{\mathbb{Z}_p}(\xi L_0)$.

3. $\mathrm{End}_{\mathbb{Z}_p}(L) = \mathrm{End}_{\mathbb{Z}_p}(M)$ if and only if $L = \alpha M$, $\alpha \in \mathbb{Q}_p^\times$, in which case L and M are homothetic and we write $[L] = [M]$.

The building for $SL_n(\mathbb{Q}_p)$

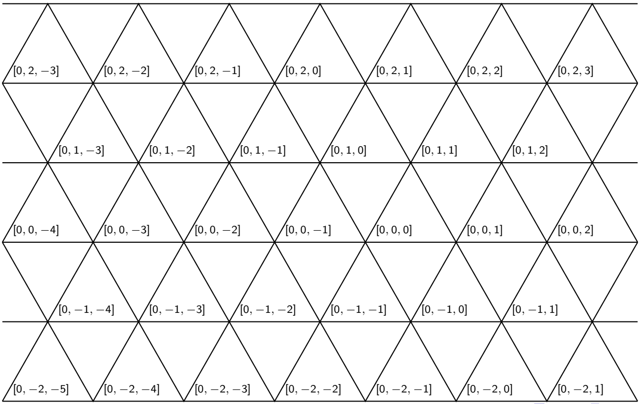
Construct the building for $SL_n(\mathbb{Q}_p)$:

- The vertices of the building correspond to homothety classes of lattices in V (or maximal orders in B).
- There is an edge between two vertices if there are lattices L_1 and L_2 in their respective homothety classes of lattices such that $pL_1 \subsetneq L_2 \subsetneq L_1$.
- The vertices of m-simplices correspond to chains of lattices $pL_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_{m+1} \subsetneq L_1$. The maximal $(n-1)$-simplices are called chambers.

The building for $SL_n(\mathbb{Q}_p)$ (cont.)

Notation. If $L = \mathbb{Z}_p p^{m_1} e_1 \oplus \mathbb{Z}_p p^{m_2} e_2 \oplus \cdots \oplus \mathbb{Z}_p p^{m_n} e_n$, then $[L] = [\mathbb{Z}_p e_1 \oplus \mathbb{Z}_p p^{m_2 - m_1} e_2 \oplus \cdots \oplus \mathbb{Z}_p p^{m_n - m_1} e_n]$, and we encode $[L] = [0, m_2 - m_1, m_3 - m_1, \ldots, m_n - m_1]$.

A piece of an apartment in the building for $SL_3(\mathbb{Q}_p)$:



Intro to tiled orders

Definition

Let $\Gamma \subset B$ be an order. We say Γ is **tiled** *if it contains a conjugate of the ring diag$(\mathbb{Z}_p, \mathbb{Z}_p, \ldots, \mathbb{Z}_p)$.*

By restricting ourselves to one apartment, we may assume diag$(\mathbb{Z}_p, \mathbb{Z}_p, \ldots, \mathbb{Z}_p) \subset \Gamma$, in which case:

1. $\Gamma = (p^{\nu_{ij}}\mathbb{Z}_p)$, where

$$\nu_{ij} + \nu_{jk} \geq \nu_{ik}, \ \ \nu_{ii} = 0 \quad \text{for all} \ \ i, j, k \leq n.$$

We define the exponent matrix $M_\Gamma := (\nu_{ij}) \in M_n(\mathbb{Z})$.

2. $\Gamma = \bigcap_{i=1}^{n} \Lambda_i$ can be written as the intersection of n maximal orders, and we can choose Λ_i to be $\mathrm{End}_{\mathbb{Z}_p}(P_i)$, where P_i is given by the ith column of Γ.

Tiled orders (example)

$$\Gamma = \begin{pmatrix} \mathbb{Z}_p & p\mathbb{Z}_p & p^2\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & p\mathbb{Z}_p \\ \mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} = \Lambda_1 \cap \Lambda_2 \cap \Lambda_3, \text{ where}$$

$$\Lambda_1 = M_3(\mathbb{Z}_p) = \mathrm{End}(\mathbb{Z}_p e_1 \oplus \mathbb{Z}_p e_2 \oplus \mathbb{Z}_p e_3),$$

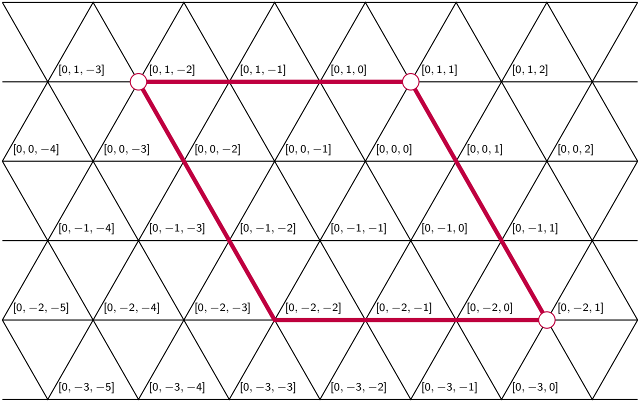
$$\Lambda_2 = \begin{pmatrix} \mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \\ p^{-1}\mathbb{Z}_p & \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ \mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} = \mathrm{End}(\mathbb{Z}_p p e_1 \oplus \mathbb{Z}_p e_2 \oplus \mathbb{Z}_p p e_3),$$

$$\Lambda_3 = \begin{pmatrix} \mathbb{Z}_p & p\mathbb{Z}_p & p^2\mathbb{Z}_p \\ p^{-1}\mathbb{Z}_p & \mathbb{Z}_p & p\mathbb{Z}_p \\ p^{-2}\mathbb{Z}_p & p^{-1}\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} = \mathrm{End}(\mathbb{Z}_p p^2 e_1 \oplus \mathbb{Z}_p p e_2 \oplus \mathbb{Z}_p e_3).$$

Convex polytopes C_Γ of Γ

$$\Gamma = \begin{pmatrix} \mathbb{Z}_p & p^2\mathbb{Z}_p & p^2\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & p^3\mathbb{Z}_p \\ p\mathbb{Z}_p & p^3\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \text{ with } M_\Gamma = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$

gives the following polytope C_Γ:



The normalizer of a tiled order

Definition

Let $\mathcal{N}(\Gamma) = \{\xi \in GL_n(\mathbb{Q}_p) \,|\, \xi\Gamma\xi^{-1} = \Gamma\}$. This is the **normalizer** *of Γ in B.*

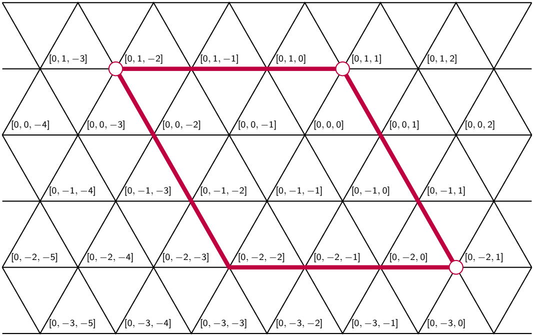
The normalizer permutes the vertices given by the columns of the exponent matrix and corresponds to symmetries of the convex polytope C_Γ. This induces a group homomorphism $\mathcal{N}(\Gamma) \to S_n$, with kernel $\mathbb{Q}_p^\times \Gamma^\times$.

Our goal is to compute $\mathcal{N}(\Gamma)/\mathbb{Q}_p^\times \Gamma^\times \hookrightarrow S_n$.

The normalizer of a tiled order (example)

$$\textit{For } M_\Gamma = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$

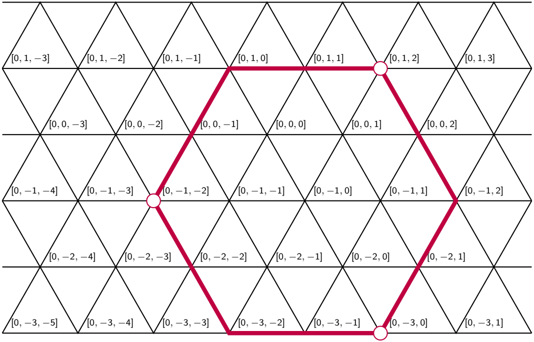
The symmetry of the polytope corresponds to a fold, and the normalizer is $\mathbb{Z}/2\mathbb{Z}$.



The normalizer of a tiled order (example)

$$\text{For} \quad M_\Gamma = \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix}.$$

The polytope has both reflections and rotations, and the normalizer corresponds to S_3.



Structural invariants

Definition (Zassenhaus, 1975)

Given $\Gamma = (p^{\nu_{ij}}\mathbb{Z}_p)$ a tiled order, the structural invariants of Γ are given by

$$m_{ijk} = \nu_{ij} + \nu_{jk} - \nu_{ik}, \text{ for } 1 \leq i, j, k \leq n.$$

Proposition (Zassenhaus, 1975)

Two tiled orders Γ with set of structural invariants $\{m_{ijk}\}$ and Γ' with set of structural invariants $\{m'_{ijk}\}$ are isomorphic if and only if there exists $\sigma \in S_n$ such that

$$m_{ijk} = m'_{\sigma(i)\sigma(j)\sigma(k)} \text{ for all } 1 \leq i, j, k \leq n.$$

Structural invariants and the normalizer

Proposition (B., 2017)

Let H be the subgroup of S_n given by $\mathcal{N}(\Gamma)/\mathbb{Q}_p^\times \Gamma^\times \cong H$. Then $\sigma \in H$ if and only if

$$m_{ijk} = m_{\sigma(i)\sigma(j)\sigma(k)} \text{ for all } 1 \leq i, j, k \leq n.$$

Moreover, we get a monomial representative of σ given by $\xi_\sigma = (p^{\nu_{i1} - \nu_{\sigma(i)\sigma(1)}} \delta_{\sigma(i)j}) \in \mathcal{N}(\Gamma)$, where δ_{ij} is the Kronecker delta.

Computing the normalizer

Given a tiled order Γ, we construct a tiled order Γ_0 such that $\mathcal{N}(\Gamma_0)/\mathbb{Q}_p^\times \Gamma_0^\times \cong \mathcal{N}(\Gamma)/\mathbb{Q}_p^\times \Gamma^\times$, where all the monomial representatives $\xi_\sigma \in \mathcal{N}(\Gamma_0)$ are given by permutation matrices.

Theorem (B., 2017)

Given a tiled order $\Gamma = (p^{\nu_{ij}})$ with structural invariants $\{m_{ijk}\}$, define $\Gamma_0 = (p^{\mu_{ij}})$, where $\mu_{ij} = \sum_{k=1}^n m_{ijk}$. Then Γ_0 is a tiled order with structural invariants $\widetilde{m}_{ijk} = n \cdot m_{ijk}$ for all $1 \leq i, j, k \leq n$, and $\sigma \in S_n$ is a symmetry of C_Γ if and only if $\mu_{ij} = \mu_{\sigma(i)\sigma(j)}$.

Example

$$M_\Gamma = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 3 & 0 & 4 & 3 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 5 & 0 \end{pmatrix} \longrightarrow M_{\Gamma_0} = \begin{pmatrix} 0 & 8 & 8 & 12 \\ 8 & 0 & 8 & 12 \\ 12 & 12 & 0 & 4 \\ 8 & 8 & 12 & 0 \end{pmatrix}$$

In M_{Γ_0}, $\nu_{12} = \nu_{21}, \nu_{13} = \nu_{23}, \nu_{14} = \nu_{24}, \nu_{31} = \nu_{32}, \nu_{41} = \nu_{42}$ so $\sigma = (12)$ is a symmetry. In fact,

$$\mathcal{N}(\Gamma)/\mathbb{Q}_p^\times \Gamma^\times \cong \mathbb{Z}/2\mathbb{Z}.$$

Quivers

Wiedemann and Roggenkamp defined the quiver of a tiled order:

- The vertices of $Q(\Gamma)$ are the (projective indecomposable) Γ-lattices $\{P_i\}_{i=1}^n$ given by the columns of Γ.
- There exists an arrow α from i to j if P_i is a direct summand in the projective cover of rad P_j. Explicitly, there exists an arrow $i \to j$ in $Q_v(\Gamma)$ if $m_{jki} > 0$ for all $k \neq i, j$, and there is an arrow $i \to i$ if $m_{iki} > 1$ for all $k \neq i$.

Theorem (Haefner, Pappacena, 2002)

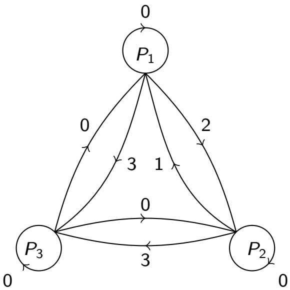
Given a tiled order Γ with $\mathcal{N}(\Gamma)/\mathbb{Q}_p^\times \Gamma^\times \cong H \leq S_n$, there is an inclusion

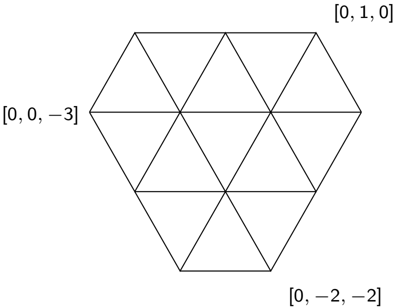
$$H \hookrightarrow Aut(Q(\Gamma)).$$

Valued quivers

To obtain a valued quiver $Q_v(\Gamma)$, we denote the value of each arrow $\alpha : i \to j$ in the quiver by $v(\alpha) = \nu_{ij}$.

Γ tiled order with exponent matrix $M_\Gamma = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$





Valued quivers (cont.)

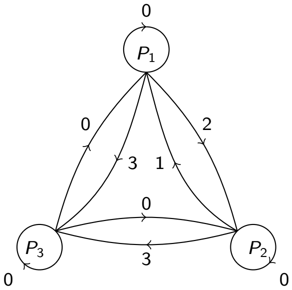
Theorem (B., 2018)

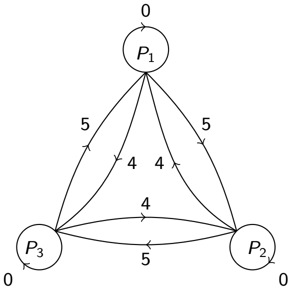
Let $\Gamma = (p^{\nu_{ij}})$ *be a tiled order and* $\Gamma_0 = (p^{\mu_{ij}})$ *as defined earlier. Then*

$$\mathcal{N}(\Gamma)/\mathbb{Q}_p^\times \Gamma \cong \mathcal{N}(\Gamma_0)/\mathbb{Q}_p^\times \Gamma_0 \cong Aut(Q_v(\Gamma_0)).$$

Valued quivers (cont.)

$$\Gamma : \begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \Gamma_0 : \begin{pmatrix} 0 & 5 & 4 \\ 4 & 0 & 5 \\ 5 & 4 & 0 \end{pmatrix}$$





A more involved example

$$\Gamma : \begin{pmatrix} 0 & 1 & 3 & 3 & 1 \\ 2 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 & 0 \end{pmatrix} \quad \longrightarrow \quad \Gamma_0 : \begin{pmatrix} 0 & 5 & 10 & 10 & 5 \\ 10 & 0 & 5 & 10 & 5 \\ 5 & 5 & 0 & 10 & 10 \\ 5 & 10 & 5 & 0 & 10 \\ 10 & 10 & 5 & 5 & 0 \end{pmatrix}$$

$$\longrightarrow \quad Q_v(\Gamma_0) : \begin{pmatrix} 0 & & 10 & 10 & \\ 10 & 0 & 5 & & \\ & 5 & 0 & 10 & 10 \\ & & 5 & 0 & 10 \\ 10 & 10 & & & 0 \end{pmatrix}$$

Thank you!