Computing Limitations to the LP Method for Sphere Packing from Non-negative Modular Forms

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The Sphere Packing Problem

What proportion of $\mathbb{R}^d$ can be covered with unit spheres?

Despite hundreds of years of study, the answer is only known in dimensions $1, 2, 3, 8, 24$. Many of these cases were solved only recently. Thomas Hales and collaborators solved the problem in $\mathbb{R}^3$ via a famously massive computational proof in 2005. The proof was finally verified using automated proof checking in 2014. [1] The solutions in dimensions 8 and 24, based on the LP method, were the 2016 result of conceptual breakthroughs by Maryna Viazovska and collaborators. [2, 3] In all other dimensions, the problem is wide open. Progress has come in the form denser sphere packings and better upper bounds.

This project shows that the LP method, currently the best approach for providing upper bounds in dimensions 4 to 36, has already been pushed close to its limit in dimensions $12, 16, 20, 28$, and 32. It will take a different approach or the discovery of denser packings to solve the sphere packing problem in these dimensions.

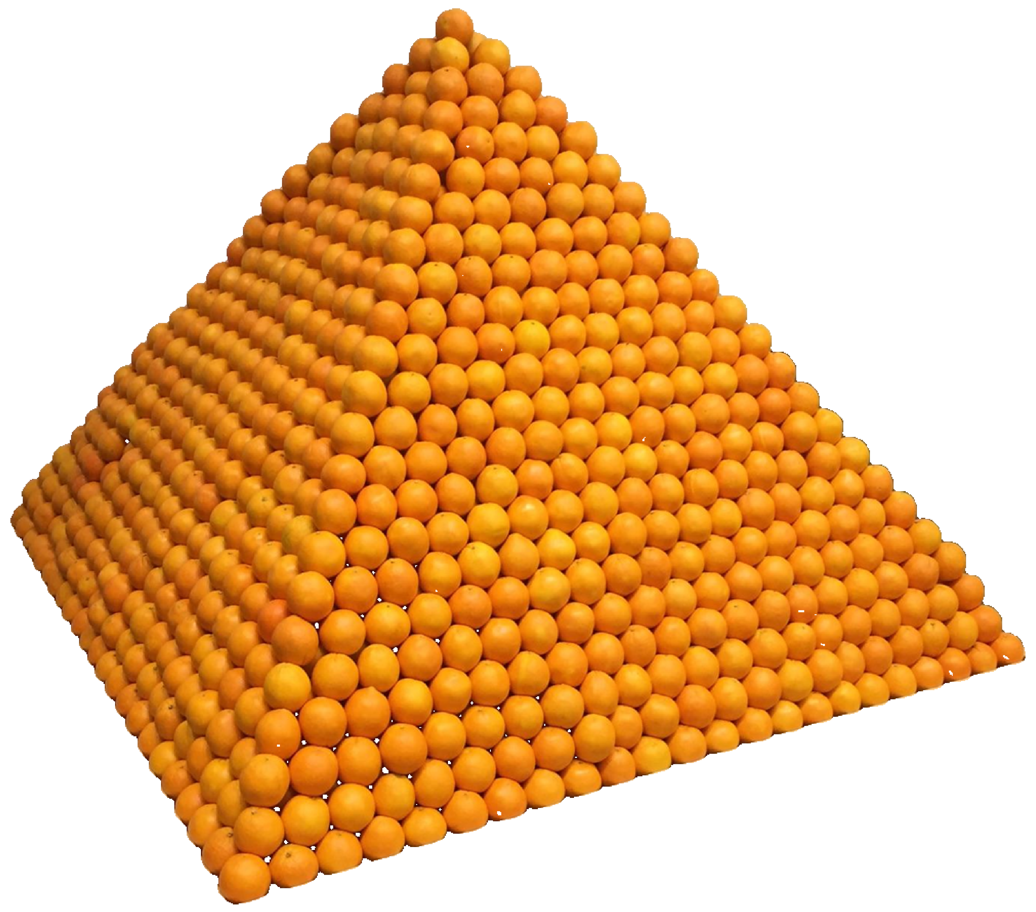

Figure 1: Approximately 5800 oranges arranged in a face-centered cubic lattice, which gives a maximal density sphere packing in dimension 3. Image from the 1967 art exhibit Soul City (Pyramid of Oranges) by Roelof Louw.

Introduction to the LP Method

The LP method, developed by Cohn and Elkies in [4], uses linear programming over spaces of functions to prove upper bounds on the density of sphere packings. It is the source of the best known upper bounds for the density of sphere packings in dimensions 4 through 36. The method was used by Viazovska in 8-dimensions [2] and Viazovska and collaborators in 24 dimensions [3] to prove that the densest possible sphere packings in these dimensions are centered on the points of the E_8 and Leech lattices.

High-level idea of LP method:

$F : \mathbb{R}^d \to \mathbb{R}$ such that $F(x) \le 0 \ \forall |x| \ge 1$, $\widehat{F}(t) \ge 0 \ \forall\, t \in \mathbb{R}^d$.

$\Downarrow$ linear function

Upper bound on density of sphere packings in $\mathbb{R}^d$.

The space of such functions is convex, so it possible to use linear programming to search for sphere packing bounds computationally.

The LP Method in More Detail

Notation:
Given $f : \mathbb{R} \to \mathbb{R}$, let $\widehat{f} : \mathbb{R} \to \mathbb{R}$ defined by

$$\widehat{f}(t) = 2\pi |t|^{1-\frac{d}{2}} \int_0^\infty f(x) J_{\frac{d}{2}-1}(2\pi x|t|) x^{d/2} dx$$

be its d-dimensional radially symmetric Fourier transform.
If we define $F : \mathbb{R}^d \to \mathbb{R}$ and $\widehat{F} : \mathbb{R}^d \to \mathbb{R}$ by $F(x) = f(|x|)$ and $\widehat{F}(t) = \widehat{f}(|t|)$, then $\widehat{F}$ is the usual Fourier transform of F.
Say f is admissible if f and $\widehat{f}$ are both $O((1+|x|)^{-n-\delta})$ for some $\delta > 0$.

Theorem 1 (Cohn-Elkies [4]).
Suppose $f : \mathbb{R} \to \mathbb{R}$ is an admissible function such that:

1. $f(x) \le 0$ for all $|x| \ge r$,
2. $\widehat{f}(t) \ge 0$ for all $t \in \mathbb{R}$,
3. $f(0) > 0$,
4. $\widehat{f}(0) > 0$.

The density of a sphere packing in $\mathbb{R}^d$ is bounded above by

$$\frac{f(0)}{\widehat{f}(0)} \cdot \frac{r^d \pi^{d/2}}{2^d \Gamma((d+2)/2)} = \frac{1}{\Gamma((d+2)/2)} \left(\sqrt[d]{\frac{f(0)}{\widehat{f}(0)}} \cdot \frac{r\sqrt{\pi}}{2} \right)^d .$$

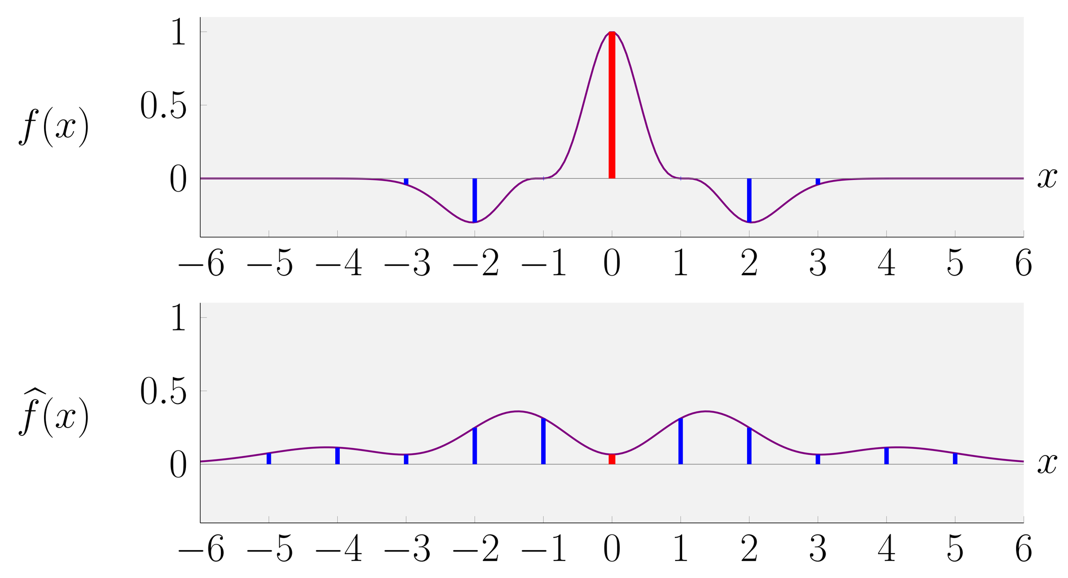

Figure 2: A typical function f and $\widehat{f}$ for the LP method. Although this example is far from optimal, the optimal function is known in dimensions, 1, 8, and 24.

Proof of Theorem 1 for Lattice Packings:
Let Λ be a lattice in $\mathbb{R}^d$ with shortest vector of length at least r and with dual lattice $\Lambda^\vee$. Let $F(x) = f(|x|)$. By our assumptions on f and Poisson summation,

$$f(0) \ge \sum_{x \in \Lambda} F(x) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^\vee} \widehat{F}(t) \ge \frac{\widehat{f}(0)}{|\Lambda|}.$$

Hence,

$$\frac{r^d \pi^{d/2}}{2^d \Gamma((d+2)/2)} \cdot \frac{f(0)}{\widehat{f}(0)} \ge \frac{r^d \pi^{d/2}}{2^d \Gamma((d+2)/2)} \frac{1}{|\Lambda|} = \left(\begin{array}{c} \text{density of sphere packing} \\ \text{centered at points of } \Lambda. \end{array} \right)$$

References

[1] Thomas Hales, Mark Adams, Gertrud Bauer, Tat Dat Dang, John Harrison, Le Truong Hoang, Cezary Kaliszyk, Victor Magron, Sean Mclaughlin, Tat Thang Nguyen, and et al. A Formal proof of the Kepler conjecture. *Forum Math. Pi*, 5:e2, 29, 2017.

[2] Maryna S. Viazovska. The sphere packing problem in dimension 8. *Ann. of Math. (2)*, 183(3):991–1015, 2016.

[3] Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, and Maryna Viazovska. The sphere packing problem in dimension 24. *Ann. of Math. (2)*, 183(3):1017–1033, 2016.

[4] Henry Cohn and Noam Elkies. New upper bounds on sphere packings. I. *Ann. of Math. (2)*, 157(2):689–714, 2003.

Key Inequality Obstructing LP Method

Notation:
For $g \in M_k(\Gamma_0(N))$ a modular form of weight k and level $\Gamma_0(N)$, let

$$\widetilde{g}(z) = g|_{w_N}(z) = \frac{i^k}{N^{k/2} z^k} g\left(-\frac{1}{Nz}\right)$$

be the dual of g under the full-level Atkin-Lehner involution.
Write the q-expansions of g and $\widetilde{g}$ as

$$g(z) = \sum_{n=0}^\infty a_n q^n, \quad \widetilde{g}(z) = \sum_{n=0}^\infty b_n q^n.$$

Theorem 2 (Cohn-T.).
Suppose $g \in M_k(\Gamma_0(N))$ is a modular form such that

- 1a. $a_n = 0$ for all $0 < n < m$,
- 1b. $a_n \ge 0$ for all $m \le n$,
- 2. $b_n \ge 0$ for all $0 \le n$,
- 3. $a_0 > 0$,

and that f satisfies the conditions of Theorem 1 with $r = \sqrt{m}$. Then,

$$\frac{f(0)}{\widehat{f}(0)} \ge \left(\frac{2}{\sqrt{N}}\right)^k \frac{b_0}{a_0}.$$

Hence,

$$\left(\begin{array}{c} \text{The optimal LP upper bound in} \\ \text{dimension } 2k \text{ from Theorem 1} \end{array} \right) \ge \frac{1}{k!} \left(\frac{m\pi}{2\sqrt{N}} \right)^k \frac{b_0}{a_0}.$$

Key to the Proof:
Replace the Poisson summation in the proof of Theorem 1 with

$$\sum_{n=0}^\infty a_n f(\sqrt{n}) = \left(\frac{2}{\sqrt{N}}\right)^k \sum_{n=0}^\infty b_n \widehat{f}\left(\frac{2\sqrt{n}}{\sqrt{N}}\right).$$

Main Result: New Obstructions to the LP Method in Dimensions 12, 16, 20, 28, and 32

The LP method is a powerful tool for bounding densities of sphere packings, but computational data suggests that the LP method does not give optimal bounds, except in dimensions 1, 8, and 24. We explain this gap by constructing explicit 'positive' modular forms which obstruct the LP method. As Figure 3 shows, our obstructions demonstrate that in dimensions which are multiples of 4, the current LP bounds are nearly best-possible for the method.

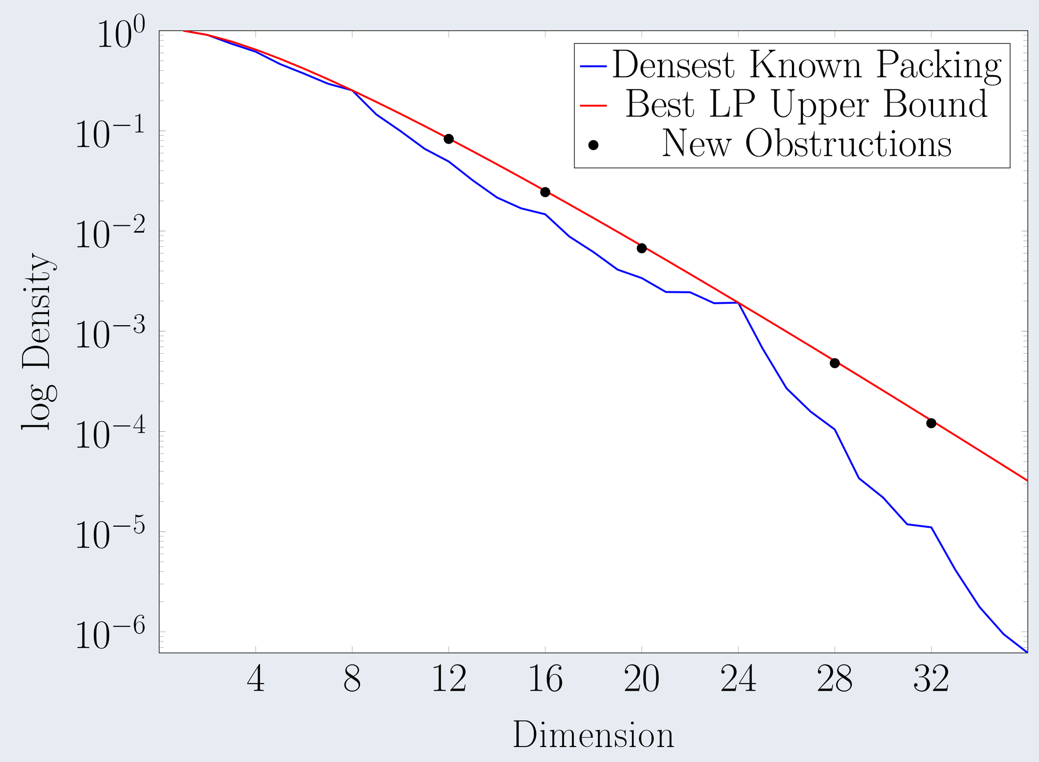

Figure 3: The upper curve is the best known linear programming bound. The lower curve is the densest sphere packing currently known. Our new obstructions show that the linear programming bound cannot be improved much.

A Procedure for Bounding the LP Method

To find obstructions to the LP method, fix N and m and compute modular forms maximizing b_0/a_0 subject to the hypotheses of Theorem 2. Repeat for many different N and m to get the best bounds.
We assume that N is not divisible by 16^2, 9^2 or p^2 for any prime $p > 3$ so that $M_k(\Gamma_0(N))$ has an eigenbasis with q-expansions in $\mathbb{Q}[[q]]$.

Procedure:

1. Compute a basis for $M_k(\Gamma_0(N))$ and duals to precision q^T.
2. Use an LP solver to find $g \in M_k(\Gamma_0(N))$ maximizing b_0 subject to:
   - 1a. $a_n = 0$ for all $0 < n < m$,
   - 1b. $a_n \ge 0$ for all $m \le n \le T$,
   - 2. $b_n \ge 0$ for all $0 \le n \le T$,
   - 3. $a_0 > 0$,
3. Check that $a_n, b_n \ge 0$ for all $n \in \mathbb{N}$ as in the next section.
4. Output b_0 if all a_n, b_n are non-negative. If not, increase T and try again. $T \approx 2 \cdot \dim M_k(\Gamma_0(N))$ usually seems to work.

Remark: Most LP solvers output solutions that only approximately satisfy the constraints, but we need exact solutions! Although solvers that give exact solutions exist, informed guesswork and linear algebra is usually a faster way to turn approximate solutions into exact solutions. Ask me how I guess the forms!

Checking Positivity of q-Expansions

To check that a modular form $g \in M_k(\Gamma_0(N))$ obstructs the LP method, we must verify that all of the coefficients of the q-expansion of g and $\tilde{g}$ are non-negative. We take the following five-step approach.

1. Write $g = g_e + g_c$, where g_e is the Eisenstein part of g,
$$g_e = e_0 + \sum_{n=1}^\infty e_n q^n \in \mathcal{E}_k(N)$$
and g_c is the cuspidal part of g,
$$g_c = \sum_{n=1}^\infty c_n q^n \in S_k(N).$$
2. Express g_e as a linear combination of Eisenstein series and use explicit formulas to get a bound of the form
$$e_n \ge A \cdot \sigma_{k-1}(n) > An^{k-1}$$
for some constant $A \in \mathbb{R}$.
3. Express g_c as a linear combination of normalized eigenforms and use Deligne's Weil bounds to get a bound of the form
$$|c_n| \le B\sigma_0(n) n^{(k-1)/2} \le Bn^{k/2}.$$
for some constant $B \in \mathbb{R}$.
4. Compare An^{k-1} to $Bn^{k/2}$ to find $Q \in \mathbb{Z}$ such that $e_n + c_n \ge 0$ for all $n \ge Q$.
5. Check that $e_n + c_n \ge 0$ for all $0 \le n < Q$ by explicit computation.

For More Information

- For a preprint or code, email ngtriant@mit.edu.
- This poster will be available at `http://www-math.mit.edu/~ngtriant/research.html`

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