Two Algorithms to Find Primes in Patterns*

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Prime Patterns

Mathematicians are interested in prime numbers, and how they can appear in patterns.

Twin Primes and Prime k-Tuples

- One example of a simple pattern is twin primes, which follow the pattern $(p, p+2)$.
- Zhang [20] recently showed that there exists a postive integer h where there are infinitely many primes in the pattern $(p, p+h)$.
- Generalizing this idea to more primes leads to **prime k-tuples** or **prime constellations**.

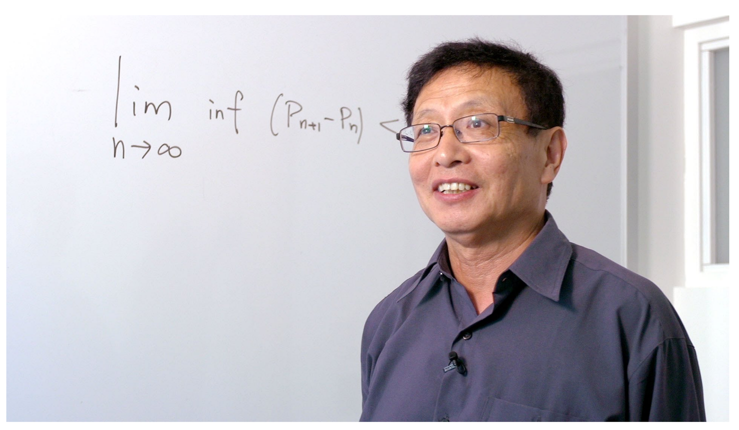
Sophie Germain Primes and Cunningham Chains

Sophie Germain was interested in the pattern $(p, 2p+1)$. Extending this idea leads to **Cunningham Chains**:

- Chains of the **first kind**: $(p, 2p+1, 4p+3, \ldots)$
- Chains of the **second kind**: $(p, 2p-1, 4p-3, \ldots)$.

Sophie Germain aos 14 anos, por Auguste Eugene Leray.

Sophie Germain



Yitang Zhang

Prime Pattern Definition

Let $k > 0$ be an integer. We define a **prime pattern** of size k as a list of linear polynomials over the integers with positive leading coefficients

$$(f_1(x), \ldots, f_k(x)).$$

Distribution of Primes in Patterns

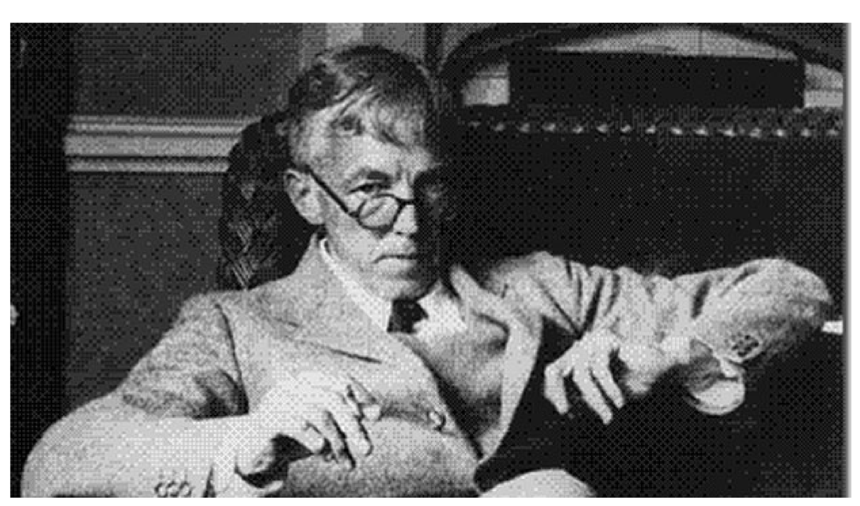
- **The Hardy-Littlewood k-tuple conjecture [9]** implies that each such pattern, with leading coefficient 1, that is **admissible**, will be satisfied by primes infinitely often.
- Further, the conjecture implies that the number of primes $\le n$ in such a pattern of length k is roughly proportional to

$$\frac{n}{(\log n)^k}.$$

- A pattern of size k is *admissible* if, for every prime $p \le k$, there is an integer x such that p does not divide any of the $f_i(x)$.
- Dickson [4] conjectured that there are infinitely many primes satisfying admissible patterns with arbitrary positive leading coefficients.
- Halberstam and Richert [8, Theorem 2.4] proved the upper bound

$$O\left(\frac{n}{(\log n)^k}\right)$$

for the number of primes $\le n$ that satisfy a pattern of length k.



G. H. Hardy

John Edensor Littlewood

The Algorithmic Problem

Given a pattern of length k, $(f_1(x), \ldots, f_k(x))$, and a bound n, find all positive integer values of x such that all the $f_i(x)$ are prime, and $\max\{f_i(x)\} \le n$.

Previous Work

- **Algorithms:**
  Günter Löh [13] and Tony Forbes [5] published partial algorithm descriptions, and used their algorithms to find various primes in patterns.
- **Complexity:**
  As far as we are aware, no complexity analysis has been published.
  All primes $\le n$ can be found, and the resulting list scanned for patterns. This takes time $O(n/\log\log n)$ using $\sqrt{n}$ space, or $O(n)$ time using roughly $n^{1/3}$ space [1, 6].
- **Computational Results and Records:**
  Record computations can be found online here:
  – http://primerecords.dk, which is maintained by Jens Kruse Andersen.
  – *The Prime Pages* at primes.utm.edu has some as well.
  – The *Online Encyclopedia of Integer Sequences*, OEIS.org, has many entries related to primes in patterns, including A001359 (twin primes), A007530 (prime quadruplets), and A005602 and A109828 (Cunningham chains).

Our New Results

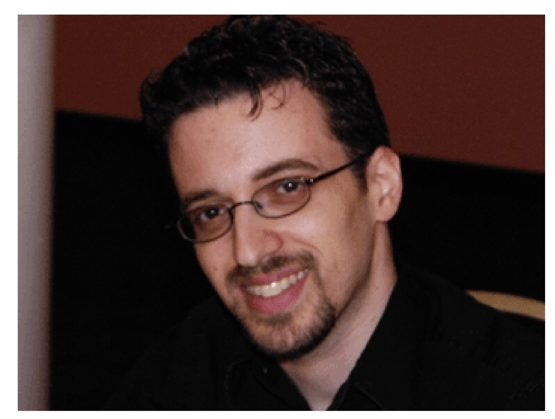
Theorem 1.

Given a pattern of length k with positive leading coefficients, and a search bound n, there is an algorithm to list all integers x such that $\max\{f_i(x)\} \le n$ and all the $f_i(x)$ are prime. This algorithm uses at most

$$O\left(\frac{nk}{(\log\log n)^k}\right) \text{ arithmetic operations (time) and } O(k\sqrt{n}) \text{ bits of space.}$$

- This algorithm extends the Atkin-Bernstein prime sieve [1] with the space-saving wheel sieve [17, 18, 19].

A. O. L. Atkin



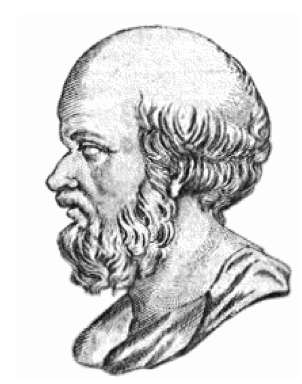
Daniel J. Bernstein

Theorem 2.

Let c be a constant with $0 < c < 1/2$. Given a pattern of length $k > 6$ with positive leading coefficients, and a search bound n, there is an algorithm to list all integers x such that $\max\{f_i(x)\} \le n$ and all the $f_i(x)$ are prime. This algorithm uses at most

$$O\left(\frac{nk}{(\log\log n)^{k-1}}\right) \text{ arithmetic operations (time) and } O(n^c) \text{ bits of space.}$$

- Due to the much smaller space use, this is a very practical algorithm.
- If we assume a conjecture due to Bach and Heulsbergen [2], we can take k as small as 3.
- This version uses the Sieve of Eratosthenes in place of the Atkin-Bernstein sieve, and supplements with base-2 pseudoprime tests [16] and the pseudosquares prime test of Lukes, Patterson, and Williams [14].



Eratosthenes of Cyrene

New Computational Results

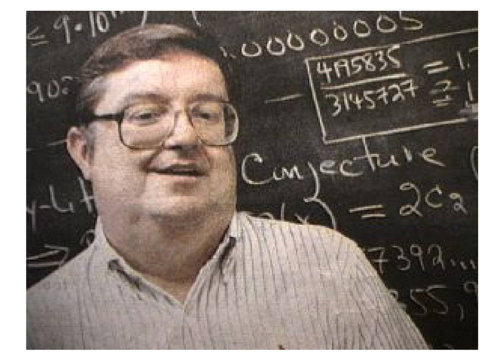
Twin Primes and Brun's Constant

Let $\pi_2(X)$ count the twin prime pairs $(p, p+2)$ with $p < X$ and $S_2(X)$ be the sum of their reciprocals. Thomas Nicely computed these functions up to $2 \cdot 10^{16}$ (See http://www.trnicely.net/#PI2X). We verified his computations and extended the results to $X = 10^{17}$.

$\pi_2(x)$ is known higher than 10^{17}, but the reciprocal sums are new.

X	$\pi_2(x)$	$S_2(X)$
$1 \cdot 10^{16}$	10304195697298	1.83048442465833932906
$2 \cdot 10^{16}$	19831847025792	1.83180806343237985727
$3 \cdot 10^{16}$	29096690339843	1.83255992186282759050
$4 \cdot 10^{16}$	38196843833352	1.83308370147757159450
$5 \cdot 10^{16}$	47177404870103	1.83348457901336613822
$6 \cdot 10^{16}$	56064358236032	1.83380868220200440399
$7 \cdot 10^{16}$	64874581322443	1.83408033035537994465
$8 \cdot 10^{16}$	73619911145552	1.83431390342560497644
$9 \cdot 10^{16}$	82309090712061	1.83451860315233433306
$10 \cdot 10^{16}$	90948839353159	1.83470066944140434160

In the last section of his PhD Thesis [12], Klyve describes how to use this information to derive bounds for Brun's constant.



Thomas Nicely

Prime Quads

A related sum involves the reciprocals of the prime tuple $(p, p+2, p+6, p+8)$. Let $\pi_4(X)$ count these tuplets up to X, and let $S_4(X)$ be the sum of their reciprocals. Thomas Nicely computed these functions up to $2 \cdot 10^{16}$. We extended this computation and partial results are in the table below. The first two lines are Thomas Nicely's own results, which we verified.

X	$\pi_4(x)$	$S_4(X)$
$1 \cdot 10^{16}$	25379433651	0.87047769123404594005
$2 \cdot 10^{16}$	46998268431	0.87048371094805250092
$3 \cdot 10^{16}$	67439513530	0.87048703104321483993
$4 \cdot 10^{16}$	87160212807	0.87048930200258802756
$5 \cdot 10^{16}$	106365371168	0.87049101694672496876
$6 \cdot 10^{16}$	125172360474	0.87049238890880442047
$7 \cdot 10^{16}$	143655957845	0.87049352884516002359
$8 \cdot 10^{16}$	161868188061	0.87049450175556017194
$9 \cdot 10^{16}$	179847459283	0.87049534891720052192
$10 \cdot 10^{16}$	197622677481	0.87049609811047504740

Cunningham Chains

We have two computational results for Cunningham chains.

- We found the smallest chain of length 15 of the first kind, and it begins with the prime

$$p = 90616\ 21195\ 84658\ 42219.$$

The next few chains of this length of the first kind are

1 13220 80067 50697 84839
1 13710 75635 40868 11919
1 23068 71734 48294 53339
1 40044 19781 72085 69169

- In 2008 Jaroslaw Wroblewski found a Cunningham Chain of length 17 of the first kind, starting with

$$p = 27\ 59832\ 93417\ 13865\ 93519,$$

and we were able to show that this is in fact the smallest such chain of that length.

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