School of Physical, Environmental and Mathematical Sciences
The University of New South Wales Canberra, Australia
Supported by Australian Research Council Discovery Project DP160100932

Robin's Inequality for 15-free Integers

Dr Thomas Morrill

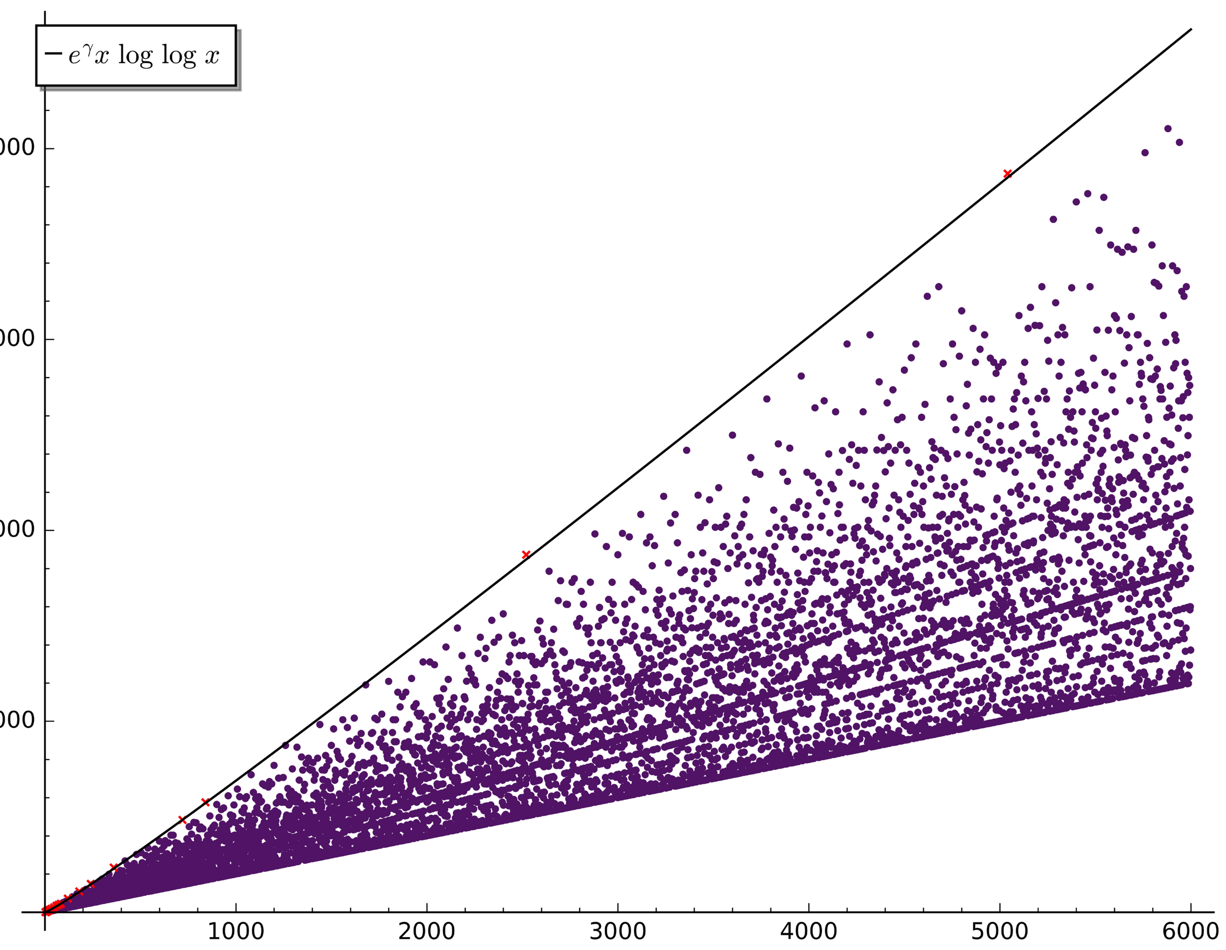
Problem Statement

In 1984, Robin [6] gave an equivalent statement of the Riemann hypothesis in terms of the sum-of-divisors function $\sigma(n)$.

> **Theorem 1 (Robin [6]):**
> The Riemann Hypothesis is true if and only if $\sigma(n) < e^{\gamma} n \log\log n$ for all $n > 5040$.

Here, γ is the Euler-Mascheroni constant. This inequality has since become known as Robin's inequality. We prove:

> **Theorem 2:**
> Robin's inequality is true for all integers $n > 5040$ which are 15-free.



There are twenty-six known exceptions to Robin's inequality, the largest of which is 5040.

History

There are many results proving Robin's inequality for infinite families of integers. We will focus on the "t-free" families of integers. We call n t-free if n is not divisible by the t^{th} power of any prime number.

- In 2007, Choie, Lichiardopol, Moree, and Solé [4] proved that Robin's inequality is true for all integers $n > 5040$ which are 5-free.
- In 2012, Solé and Planat [7] proved that Robin's inequality is true for all integers $n > 5040$ which are 7-free.
- In 2015, Broughan and Trudgian [2] proved that Robin's inequality is true for all integers $n > 5040$ which are 11-free.

Additionally, in 2006, Briggs [1] computationally verified Robin's inequality for integers $5040 \le n \le 10^{10^{10}}$.

Citations

[1] Keith Briggs. Abundant numbers and the Riemann hypothesis. *Experiment. Math.*, 15(2):251–256, 2006.
[2] Kevin A. Broughan and Tim Trudgian. Robin's inequality for 11-free integers. *Integers*, 15:Paper No. A12, 5, 2015.
[3] Jan Büthe. Estimating π(x) and related functions under partial RH assumptions. *Math. Comp.*, 85(301):2483–2498, 2016.
[4] YoungJu Choie, Nicolas Lichiardopol, Pieter Moree, and Patrick Solé. On Robin's criterion for the Riemann hypothesis. *J. Théor. Nombres Bordeaux*, 19(2):357–372, 2007.
[5] Pierre Dusart. Explicit estimates of some functions over primes. *Ramanujan J.*, 45(1):227–251, 2018.
[6] G. Robin. Grandes valeurs de la fonction somme des diviseurs et hypoth`ese de Riemann. *J. Math. Pures Appl. (9)* , 63(2):187–213, 1984.
[7] Patrick Solé and Michel Planat. The Robin inequality for 7-free integers. *Integers*, 12(2):301–309, 2012.

Method

Since we may write the sum-of-divisors function as

$$\sigma(n) = \prod_{p^a | n} (1 + p + \cdots + p^a)$$

then one can verify that for t-free n,

$$\sigma(n) \le \Psi_t(n) := n \prod_{p|n} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{t-1}}\right)$$

We then use the following lemma from Solé and Planat.

> **Lemma (Solé, Planat [7]):**
> Let $R_t(n) = \frac{\Psi_t(n)}{n \log\log n}$ and let $N_n = \prod_{k=1}^{n} p_k$ denote the nth primorial number.
> If $R_t(N_n) < e^{\gamma}$, and $n \ge 2263$, then for all $m > N_n$ we have $R_t(m) < e^{\gamma}$.

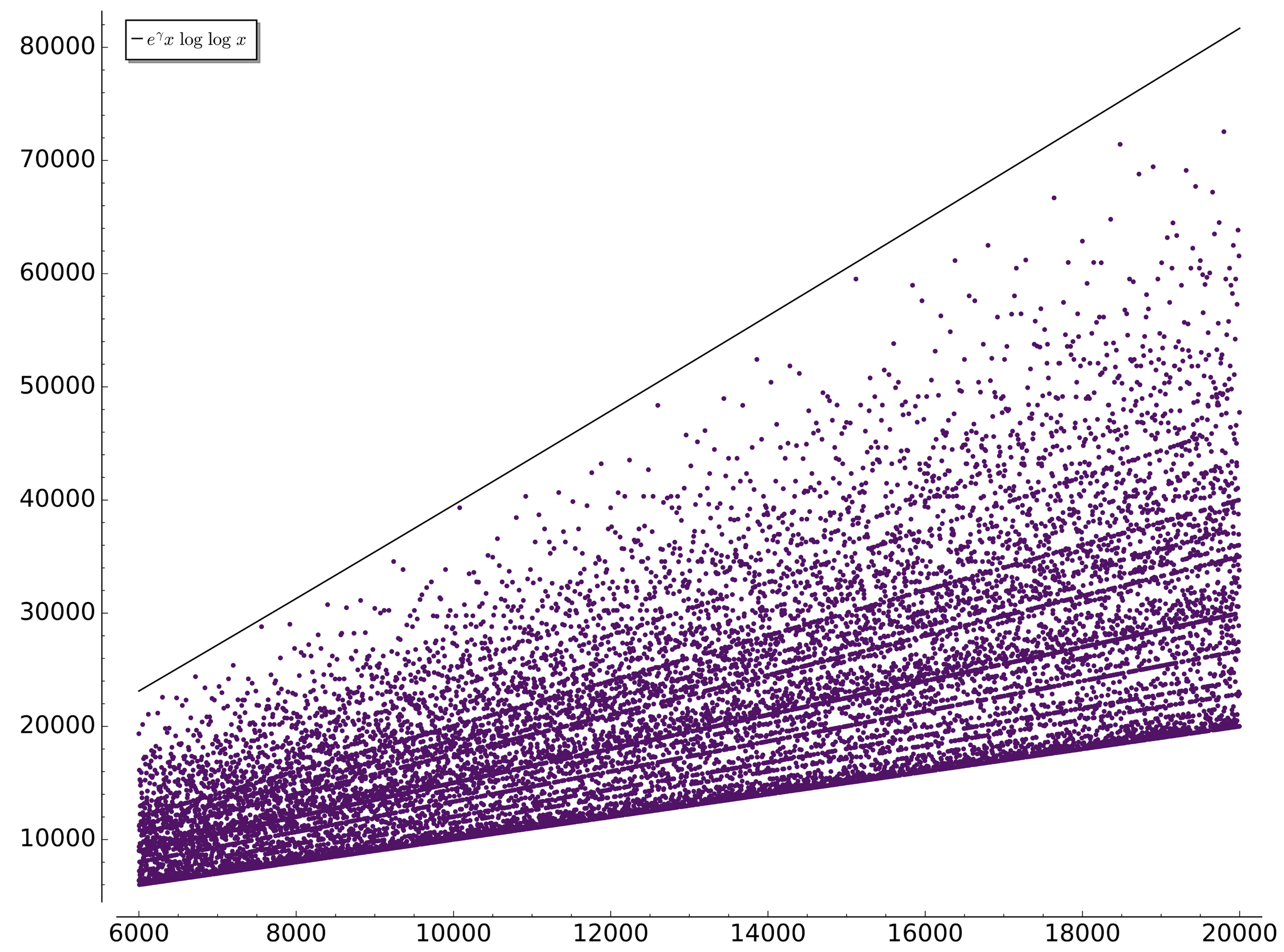
Using up-to-date bounds from Büthe [3] and Dusart [5], we find that

$$R_t(N_n) \le e^{\gamma} \frac{\exp\left(\frac{2}{p_n}\right) \log(p_n) \left(1 + \frac{0.006}{\log^2 p_n}\right)}{\zeta(t) \log\left(p_n - \frac{\sqrt{p_n} \log p_n}{8\pi}\right)}$$

where for $n \ge 688383$,

$$n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2.1}{\log n}\right) \le p_n \le n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n}\right)$$

Thus, we wish to find the smallest integer n so that $R_t(N_n) < e^{\gamma}$, from which it follows that Robin's inequality is true for all t-free integers $m > N_n$. So long as $N_n < 10^{10^{10}}$, then the inequality has already been verified for t-free integers less than N_n by Briggs [1].



Demonstration of Robin's inequality for $6000 < n < 20000$.

Conclusion

By calculation, we have that $R_{15}(N_{91431092}) < e^{\gamma}$, and that Robin's inequality holds for all 15-free $5040 < m < N_{91431092}$. Since the smallest integer which is not 15-free is 32768, we may give an equivalent form of the Riemann hypothesis:

> **Corollary to Theorem 2**:
> The Riemann Hypothesis is true if and only if Robin's inequality holds for all positive n which are divisible by the 15^{th} power of some prime.

When we attempt to prove Robin's inequality for 16-free integers, we compute that $R_{16}(N_{4118427007}) < e^{\gamma}$. However, $N_{4118427007} \sim 10^{10^{10.638}}$, meaning that there are 16-free integers between $N_{4118427007}$ and $10^{10^{10}}$ for which we have not verified Robin's inequality. Our results may be improved upon by extending Briggs' results past $10^{10^{10}}$, or by improving the known bounds [3] [5] on functions relating to the n^{th} prime number.

181279