Hidden Forest: Discovering Invisible Squares

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Introduction

Imagine you are standing in front of a forest, with lined-up trees. Each row and column are occupied by some infinitely thin trees. We consider each tree as a point. Behind visible trees, there must exist some trees that are blocked by these visible trees. In this poster, we use a $\mathbb{Z} \times \mathbb{Z}$ integer plane to represent the forest.

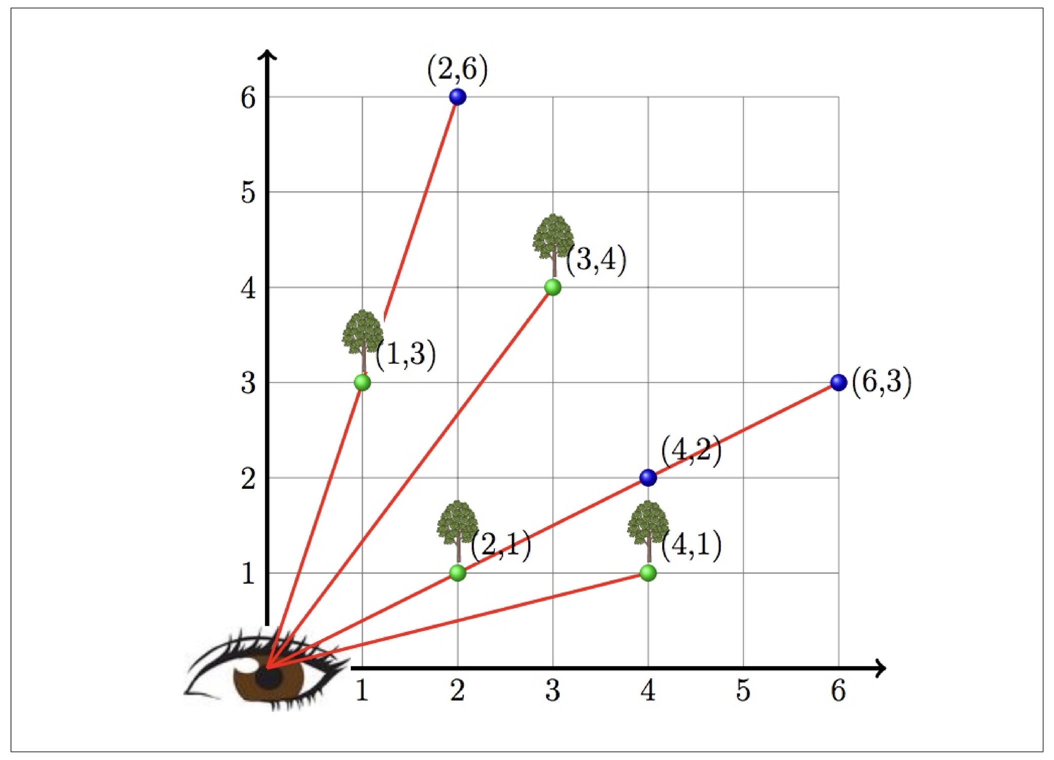


Figure 1: Visible and invisible points under different lines of sights, source: Goodrich et al.

By using the Chinese Remainder Theorem algorithm, the coordinates of the invisible squares can be found. Given standing at the point of origin, there exists an invisible square. First, we worked on finding the influence of the observers movement on the invisible square. By adjusting the location of the observer along x-axis and y-axis, the original invisible squares movement is based on the path of the observer. Second, we determined the existence of a common invisible point and then a common invisible square to three points. Our posters examples are based on three points: (0,0), (1,0), and (0,1).

Prime Matrix

Let $\{p_1, p_2, ..., p_{n^2}\}$ be the the set of the first n^2 primes. Construct an $n \times n$ matrix with these primes by filling row i with the n primes $p_{(i-1)n+1}$ through $p_{(i-1)n+n}$ for each $1 \leq i \leq n$ to yield the following:

$$P_n = \begin{bmatrix} p_1 & p_2 & \cdots & p_j & \cdots & p_n \\ p_{n+1} & p_{n+2} & \cdots & p_{n+j} & \cdots & p_{2n} \\ p_{2n+1} & p_{2n+2} & \cdots & p_{2n+j} & \cdots & p_{3n} \\ \vdots & \vdots & & \vdots & & \vdots \\ p_{(i-1)n+1} & p_{(i-1)n+2} & \cdots & p_{(i-1)n+j} & \cdots & p_{(i-1)n+n} \\ \vdots & \vdots & & \vdots & & \vdots \\ p_{(n-1)n+1} & p_{(n-1)n+2} & \cdots & p_{(n-1)n+j} & \cdots & p_{n^2} \end{bmatrix}$$

Chinese Remainder Theorem - algorithm

We used the Chinese Remainder Theorem algorithm to construct a $n \times n$ sized hidden forest. First, let R_i and C_j represent products of row i and column j, then we will have:

$$R_i = \prod_{k=1}^{n} P_{(i-1)n+k} \qquad C_i = \prod_{k=0}^{n-1} P_{kn+j}$$

The row products are pairwise relatively prime because they share no common primes. Similarly, the column products are pairwise relatively prime. Consider the following pair of systems of linear congruence:

$$\begin{cases} x+1 \equiv 0 \pmod{R_1} \\ x+2 \equiv 0 \pmod{R_2} \\ \vdots \\ x+n \equiv 0 \pmod{R_n} \end{cases} \qquad \begin{cases} y+1 \equiv 0 \pmod{C_1} \\ y+2 \equiv 0 \pmod{C_2} \\ \vdots \\ y+n \equiv 0 \pmod{C_n} \end{cases}$$

Common Invisible Square to Three Points

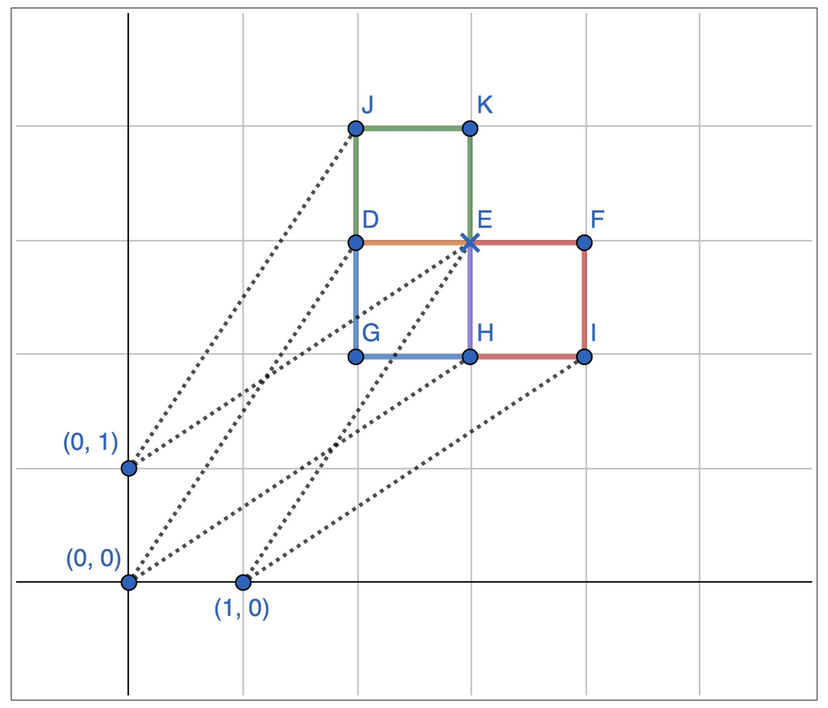
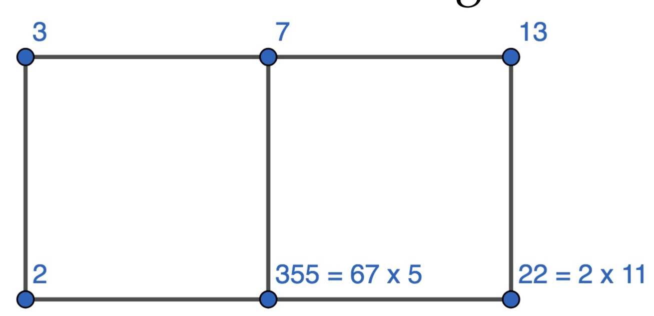


Figure 2: Common Invisible Point to Points (0,0), (1,0), and (0,1)

First, construct a 3×2 prime matrix: $P_2 = \begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 11 & 13 \end{bmatrix}$ Then, we will have:

$$\begin{cases} x+1 \equiv 0 \pmod{6} \\ x+2 \equiv 0 \pmod{35} \\ x+3 \equiv 0 \pmod{143} \end{cases} \quad \begin{cases} y+1 \equiv 0 \pmod{110} \\ y+2 \equiv 0 \pmod{273} \end{cases} \Rightarrow \begin{cases} x = 11723 \\ y = 7369 \end{cases}$$

D, E, F, G, H and I are six points that are invisible to (0,0). Then we can construct a GCD-grid as follows:



$$D = (11724, 7371) \quad E = (11725, 7371)$$
$$F = (11726, 7371) \quad G = (11724, 7370)$$
$$H = (11725, 7370) \quad I = (11726, 7370)$$

E, F, H, and I are four points that are invisible to (1,0). Square-$EFHI$ is the common invisible square to both (0,0) and (1,0); Square-$JKDE$ is the common invisible square to both (0,0) and (0,1). Hence, E is the common invisible point to (0,0), (1,0), and (0,1).

We constructed a graph based on a 4×4 matrix to find a common invisible square to (0,0), (1,0), and (0,1):

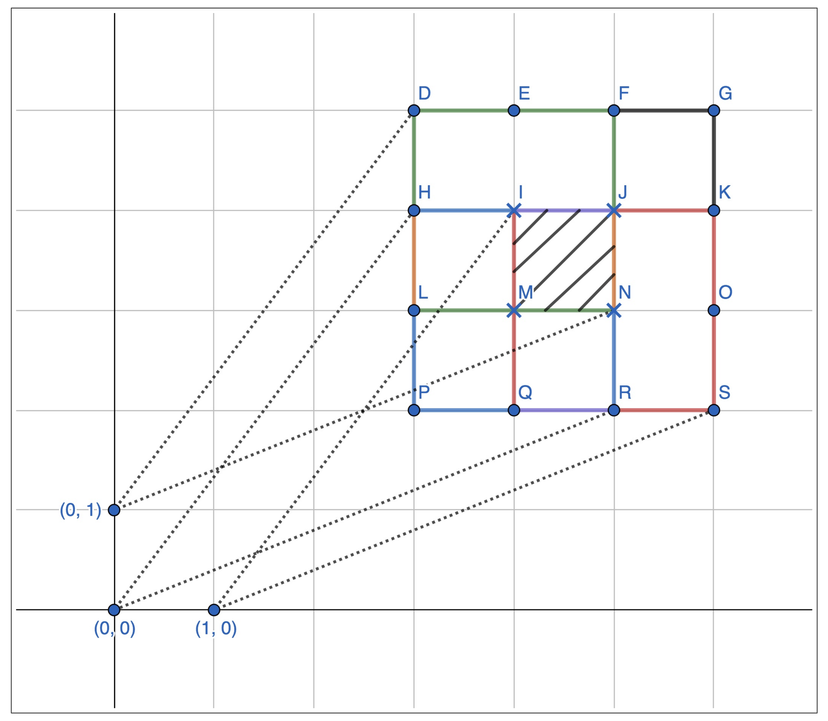


Figure 3: Common Invisible Square to Points (0,0), (1,0), and (0,1)

Square-$IJMN$ is the common invisible square to points (0,0), (1,0), and (0,1).

Future Study

The Chinese Remainder Theorem algorithm guarantees the existence of invisible squares. However, these squares might not be the closest one to the observer. One further step could be finding the closest square to the origin. In addition, future study could focus on finding the common invisible square to n points, where n is greater or equal to 3.

References

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