Sampling totally real S_4-fields

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Abstract

In 1984, Cohen and Lenstra produced empirical results on the distribution of class groups over quadratic fields. Their work motivated the following question.

Q: How frequently does a given abelian group appear as the class group of a number field?

In order to test these heuristics over arbitrary number fields, we need means for producing empirical results. This poster describes a method to sample totally real S_4-fields following Bhargava's parameterization of quartic rings [3].

Empirical Methods

Set-Up

Order the number fields of a fixed degree and Galois closure according to their "size" i.e. absolute discriminant, conductor or height. Let $X > 0$ be a bound and consider

(a) a distribution (e.g. class groups) over the finite set of number fields with size bounded by X.

(b) the asymptotic behavior of this distribution as we let the bound $X \to \infty$.

Main Problem: The difficulty in empirically testing heuristics on asymptotics lies in the fact that the distribution may be slow to converge based on the bound X.

Example: Real Quadratic Fields

Let $\mathbb{Q}(\sqrt{D})$ be a real quadratic field $(D > 0)$. Denote the class group $\mathrm{Cl}(K)$ and narrow class group $\mathrm{Cl}^+(K)$. We have the following result between these groups [4].

Theorem *Asymptotically, 100% of real quadratic fields have*

$$\mathrm{Cl}^+(K) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Cl}(K).$$

Magma can enumerate all real quadratic fields with discriminant $D < 10^9$ and calculate this relation in approximately 2 hours. For these fields, only 66% of them have this relationship.

Sampling vs. Enumeration

There are two methods for empirically testing heuristics:

1. **Enumeration.** We can enumerate all number fields up to a bound X. While this is very accurate, it can take a while for the distributions to converge based on X.
2. **Sampling.** We can randomly sample from all number fields bounded by X. While this is difficult to do, it can be done at a larger bound X relative to enumeration.

Sampling Totally Real S_4-Fields

My research focuses on developing Cohen-Lenstra type heuristics for the narrow class group. Define an S_n**-field** to be a degree n extension of $\mathbb{Q}$ with Galois closure S_n. I study the relationship between the class group and narrow class group over S_n-fields where n is even. This poster describes a method for sampling totally real S_4-fields in order to empirically test my heuristics. We do this by using Bhargava's parameterization of quartic rings [3].

Parameterization of Quartic Rings

Define a **quartic ring** to be a commutative ring that is a free $\mathbb{Z}$-module of rank 4. For example

$\mathbb{Z} \oplus \mathbb{Z}(\sqrt[3]{2})$ Orders/maximal orders in a quartic field $K/\mathbb{Q}$ $\mathbb{Z}[x]/(x^4)$

In his Ph.D thesis Manjul Bhargava established a correspondence between quartic rings and pairs A, B of integral ternary quadratic forms. The forms A, B can be represented by two 3×3 symmetric matrices

$$2 \cdot (A, B) := \left(\begin{pmatrix} 2a_{11} & a_{12} & a_{13} \\ a_{12} & 2a_{22} & a_{23} \\ a_{13} & a_{23} & 2a_{33} \end{pmatrix}, \begin{pmatrix} 2b_{11} & b_{12} & b_{13} \\ b_{12} & 2b_{22} & b_{23} \\ b_{13} & b_{23} & 2b_{33} \end{pmatrix} \right) \quad a_{ij}, b_{ij} \in \mathbb{Z},$$

which are unique up to an action of $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$. To describe this action, let $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ and $t \in \mathrm{SL}_3(\mathbb{Z})$. The matrices s, t act by

$$s \cdot (A, B) := (aA + bB, cA + dB) \qquad t \cdot (A, B) := (tAt^t, tBt^t).$$

These actions commute so for a pair $(s, t) \in \mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$ then $(s, t) \cdot (A, B)$ is well-defined. Therefore, by writing down a pair of integer matrices (A, B) we specify a quartic ring that is unique up to the action of $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$.

Sampling Algorithm (Height)

The current implementation of my algorithm is in Magma. It should be available soon on Github.

Input

- N: an integer for the sample size.
- X: a positive integer for the bound on the height of the coefficients of (A, B).

Algorithm

First, we produce a pair of matrices (A, B) by selecting 12 random integers $a_{i,j}, b_{i,j}$ with bounded absolute value $|a_{i,j}|, |b_{i,j}| < X$. Let R be the quartic ring corresponding to the pair (A, B). We now run four distinct tests on the pair (A, B) to guarantee that it is a unique reduced (in terms of the action) representative for the ring of integers of a totally real S_4-field. These tests are as follows.

1. Testing if the pair (A, B) representing R is a unique reduced representative in terms of the action of $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$.
2. Testing to see if R is irreducible i.e. $R \neq \mathbb{Z} \oplus \mathbb{Z}(\sqrt[3]{2})$.
3. Testing to see if R is maximal i.e. $R \neq \mathcal{O}$ for a non-maximal order $\mathcal{O}$ in a quartic number field.
4. Testing to see if the ring R is inside a totally real S_4-field.

These four tests can be run in any order. The fastest implementation is done when the tests are done in the order $3 \to 2 \to 1 \to 4$. If the pair (A, B) passes all four tests then we add the corresponding S_4-field to a list L. We then repeat the process and keep adding non-duplicate fields to the list L until it has length N. We then return the list L.

Results/Variants

Results

My research on the 2-ranks of the class group and narrow class group yielded the following heuristic for S_4-fields.

Conjecture (B). *Let K range across totally real S_4-fields ordered by absolute discriminant. Then*

$\mathrm{rk}_2\,\mathrm{Cl}^+(K) - \mathrm{rk}_2\,\mathrm{Cl}(K)$	$k = 0$	$k = 1$	$k = 2$
Asymptotic Density	25.87%	56.39%	17.72%

The largest tables of totally real fields available can be found on the LMFDB [5] or at John Voight's webpage. These tables enumerate all totally real quartic fields up to discriminant $|D| < 10^9$.

By running the algorithm overnight, we found a sample of 20,000 fields with discriminant $|D| < 10^{20}$. The table below shows a comparison of the difference k in 2-ranks in this sample with two similarly sized samples taken from the tables with $|D| < 10^7$ and $|D| < 10^9$ respectively.

k	$\lvert D\rvert < 10^7$	$\lvert D\rvert < 10^9$	$\lvert D\rvert < 10^{20}$	Predicted
$k = 0$	.2938	.2831	.2535	.2587
$k = 1$	.5469	.5531	.5672	.5639
$k = 2$	.1592	.1636	.1792	.1772

Although we are sampling fields based on the height of the coefficients of (A, B), we expect that this roughly corresponds to sampling fields based on discriminant.

Variants (Discriminant)

We also have a version of the algorithm that samples totally real S_4-fields with bounded discriminant. This version is based on the work of [6]. The main distinction between these algorithms are:

1. A different fundamental domain is used for testing whether a pair (A, B) was reduced with respect to the action of $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$.
2. The bounds on the coefficients $a_{i,j}, b_{i,j}$ were done iteratively and depend on the previous values of $a_{i,j}, b_{i,j}$.

This version of the algorithm runs much slower and the bounds on $a_{i,j}, b_{i,j}$ are several orders of magnitude larger.

References

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