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Integration on Riemann surfaces - Computation of period matrices

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Basic definitions

Let X be a compact Riemann surface of genus g, $\{\omega_i\}_{1\le i\le g}$ a basis of the holomorphic differentials $\Omega^1(X)$ and $\{\, c_j \,\}_{1\le j\le 2g}$ a basis of the homology group $H_1(X,\mathbb{Z})$. Then, we call

$$C = \big(\int_{c_j} \omega_i \big)_{i,j} \in \mathbb{C}^{g\times 2g} \quad \text{a (big) period matrix of } X.$$

The *Jacobian* of X is the complex torus $\mathrm{Jac}(X) := \mathbb{C}^g/\Lambda$, where Λ is the lattice generated by the columns of C.

The *Abel-Jacobi map* on X, w.r.t. $P_0 \in X$, is given by

$$\mathcal{A} : X \to \mathrm{Jac}(X)\,,\, P \mapsto \Big(\int_{P_0}^{P} \omega_1, \ldots, \int_{P_0}^{P} \omega_g \Big)^T \mod \Lambda\,.$$

In case of a *canonical homology basis* $\{\, a_j, b_j \,\}_{1\le j\le g}$, i.e. a basis satisfying $a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad a_j \circ b_k = \delta_{jk},$ we call

$$\mathcal{R} = A^{-1}B \in \mathfrak{H}_g \subset \mathbb{C}^{g\times g} \quad \text{a (small) period matrix of } X\,,$$

where $A = \big(\int_{a_i} \omega_j\big)_{1\le i,j\le g}$ and $B = \big(\int_{b_i} \omega_j\big)_{1\le i,j\le g}$.

General case: Goals

Fast and rigorous computation of $\mathcal{R}$ and $\mathcal{A}$ to high precision for compact Riemann surfaces given by polynomials $f \in \mathbb{Q}[x,y]$ using Magma. Following a mixed approach, symbolic and numerical methods are being used.

Existing work

- General case: Maple (Deconinck/Patterson/van Hoeij), Python/Sage (Swierczewski), Matlab (Frauendiener/Klein)
- Hyperelliptic case: Magma (van Wamelen), pari/gp (Molin), Matlab (Frauendiener/Klein)

Applications in number theory

- Computation of theta functions
- Canonical heights: local heights at archimedean places
- Computation of the real period of the Jacobian (BSD conjecture)
- Computation of endomorphisms and isogenies of the Jacobian
- Numerous other applications in physics (e.g. integrable PDEs)

Setup and notation

Let $f \in \mathbb{Q}[x,y]$ be geometrically irreducible. Denote by

- X the Riemann surface of genus g associated to the compactified, desingularized curve $\mathcal{C}_f$ defined by f,
- $y(x)$ an N-sheeted algebraic covering of $\mathbb{P}^1$ defined by f, where N is the degree of f in y,
- $\mathfrak{B} = \{\, x \in \mathbb{P}^1(\mathbb{C}) \mid \#y(x) < N \,\}, \quad \mathfrak{b} = \#\mathfrak{B}$.

Outline of the algorithm

Picking up the approach of [1], we can summarize the algorithm in the following steps:

- Compute a basis $\omega_1, \ldots, \omega_g$ of $\Omega^1(X)$ $\quad\to\quad$ using Magma's function fields
- Compute generators $\gamma_1, \ldots, \gamma_{\mathfrak{b}}$ of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \mathfrak{B})$ $\quad\to\quad$ see Stefan Hellbusch's poster
- Compute parameters for DE-integration $D, n, h, \tau, T = \{t_k\}_{|k|\le n}$.
- Analytically continue $y(x)$ along each γ_i in order to determine
  $\to$ the local monodromy σ_{γ_i} of γ_i, $\quad\to\quad$ see Stefan Hellbusch's poster
  $\to$ the lifts $\tilde{\gamma}_i$, represented by the fibers above $\gamma_i(t)$ for $t \in T$.
- Compute a symplectic transformation S and cycles $c_1, \ldots, c_{2g}$ generating $H_1(X,\mathbb{Z})$.
- Compute a period matrix C using DE-integration.
- Obtain a small period matrix $\mathcal{R} = A^{-1}B$ with $(\,A\ B\,) = C \cdot S$.

Continuing the fibers along paths

Suppose we know the ordered fiber $y(x_1)$ above $x_1 = \gamma(t_1)$. How do we find $y(x_2) = y(\gamma(t_2))$?

- Idea: Use the values in $y(x_1)$ as approximations of the N simple roots of the univariate polynomial $f(x_2, y) \in \mathbb{C}[y]$.
- Use efficient root approximation methods to determine the ordered fiber $y(x_2)$ to desired precision, while employing a 'divide & conquer' strategy.

Analytic continuation via simultaneous root approximation methods

Let $p \in \mathbb{C}[z]$ be a squarefree monic polynomial of degree $N \ge 2$. Suppose $z_1^{(0)}, \ldots, z_N^{(0)}$ are initial approximations of the zeros of p.

Under initial conditions that only depend on p and the $z_j^{(0)}$, the *Durand-Kerner* method given by the formula

$$z_i^{(m+1)} = z_i^{(m)} - p(z_i^{(m)}) / \prod_{j\ne i} (z_i^{(m)} - z_j^{(m)}) \quad (m \ge 0)$$

converges quadratically, providing explicit error bounds in each step.

Canonical homology basis

Taking the local monodromies as input, the Tretkoff algorithm computes a canonical homology basis, purely combinatorially.

The output consists of a generating set of cycles $\{\, c_j \,\}_{1\le j\le 2g}$ for $H_1(X,\mathbb{Z})$ and a symplectic transformation $S \in \mathrm{GL}(2g,\mathbb{Z})$.

The cycles are given as finite sums $c_j = \sum_k \tilde{\gamma}_k$ of lifted paths.

Integration of holomorphic differentials

Let $\omega \in \Omega^1(X)$ be represented as $h(x,y)\,\mathrm{d}x$, where $h \in \mathbb{C}(X)$ and $\tilde{\gamma} \in H_1(X,\mathbb{Z})$. In order to compute $\mathcal{R}$ and $\mathcal{A}$, we have to compute integrals of the form

$$\int_{\tilde{\gamma}} \omega = \int_{-1}^{1} h(\gamma(t), \tilde{\gamma}(t))\gamma'(t)\,\mathrm{d}t\,.$$

Theorem: Double-exponential integration [2]

Suppose $g :]a,b[\to \mathbb{C}$ admits an analytic continuation to $\Delta_\tau = \{\, \tanh(\sinh(\mathbb{R} \pm it)) \mid t < \tau \,\}$ for some $0 < \tau < \frac{\pi}{2}$ and that $|g| < M$ on Δ_τ. Then for all $D > 0$ there exist $n, h > 0$ such that

$$\Big| \int_a^b g(t)\,\mathrm{d}t - \sum_{k=-n}^{n} g(t_k)\mathrm{d}t_k \Big| \le e^{-D}\,,$$

where $n \sim \frac{D \log(D)}{2\pi\tau}$ and $t_k = \frac{b+a}{2} + \frac{b-a}{2}\tanh(\sinh(kh))$.

Advantages of the DE-integration

- For prescribed precision, integration parameters can be computed prior to the computation.
- Taking advantage of the integrand's holomorphicity makes integration rigorous.
- Suitable for arbitrary precision integration due to fast computation of integration parameters.

Timings

Computation of the period matrix associated to the algebraic curves given by

- $f_k = (x+y)^{k-1} + x^k y^2 + 1$ up to 20 significant digits.

k	2	3	4	5	6	7	8	9	10
g	1	2	6	10	14	21	28	35	45
t_{maple}	2.1s	6s	39s	2m 10s	error	6m 45s	12m 58s	-	error
$t_{\text{self},g}$	0.7s	1.8s	8.6s	33s	1m 29s	4m 45s	11m 33s	27m 25s	57m 27s

- $f_k = y^N - (x^d + x^{d-1} + \cdots + x + 1)$ up to 200 significant digits.

(N,d)	(2,3)	(2,5)	(3,5)	(5,5)	(7,5)	(7,8)	(11,8)	(11,11)	(11,21)	(31,21)
g	1	2	4	6	12	21	35	45	100	300
t_{maple}	1m 46s	3m 11s	6m 51s	22m 24s	1h 46m	2h 52m	-	-	-	-
$t_{\text{self},g}$	6s	15s	23s	36s	1m 9s	3m 20s	6m 20s	11m 20s	1h 8m	-
$t_{\text{self},s}$	1.2s	2.6s	1.6	2.1s	3s	8.7s	13.3s	26s	1m 50s	7m 29s

Here, $t_{\text{self},g}$ refers to the general algorithm and $t_{\text{self},s}$ to the special algorithm for the superelliptic case. Computations were done on an Intel Xeon(R) CPU E3-1275 V2 3.50GHz processor.

Outlook (work in progress)

- Abel-Jacobi map: Need to evaluate the integrand at above branch points $\to$ approximations using Puiseux series
- Integration: Need to 'bound' holomorphic differentials $\to$ choice of M
- Integration: Value of τ strongly dependent on the constructed paths, especially on the radii of arcs/circles $\to$ optimization

Special case: superelliptic curves

A superelliptic curve over $\mathbb{C}$ can be given by an affine model of the form

$$\mathcal{C} : \; y^N = f(x) = \prod_{l=1}^{d} (x - x_l)\;,$$

where $f \in \mathbb{C}[x]$ is squarefree, $N \ge 2$ and $d \ge 3$.

Superelliptic case: Goals

The following approach was developed by Pascal Molin for hyperelliptic curves. We extend the corresponding algorithm to the larger class of superelliptic curves. This is joint work in progress with Pascal Molin.

Advantages in the superelliptic case

- As $\mathcal{C}$ defines a cyclic covering of $\mathbb{P}^1$, changing sheets corresponds to multiplication by a power of $\zeta := e^{\frac{2\pi i}{N}}$ $\quad\to\quad$ local monodromy and analytic continuation easy
- Canonical basis for the space of holomorphic differentials (see Proposition below)
- Integration along line segments between branch points is possible
  $\to$ less integrals + can avoid arcs/circles
- Homology basis and intersection matrix are determined by the formula used for computing the periods (see Theorem below) $\quad\to\quad$ easily obtain a period matrix C

Proposition: Holomorphic differentials

A basis of $\Omega^1_{\mathcal{C}}$ is given by the differentials

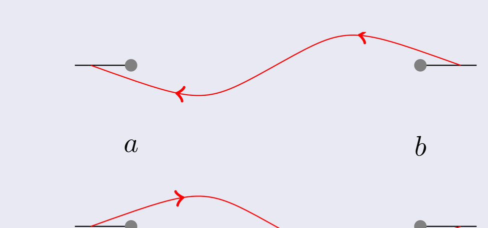
$$\mathcal{D} = \Big\{ \frac{x^i}{y^j}\mathrm{d}x \mid -Ni + jd - \gcd(N,d) \ge 0,\, 0 \le i \le d-2,\, 1 \le j \le N-1 \Big\}.$$

Theorem: Computation of periods

Let $a = x_1, b = x_2$ be branch points and and take a differential $\omega = x^i y^{-j}\mathrm{d}x \in \Omega_{\mathcal{C}}$. Then the integral computed with the formula

$$\int_{\alpha^{(0)}} \omega = \Big(\frac{b-a}{2}\Big)^{i+1-\frac{dj}{N}} \Big(\sqrt{\zeta^j} - \sqrt{\zeta^{-j}}\Big) \int_{\mathbb{R}} \frac{\Big(\tanh(t) + \frac{b+a}{b-a}\Big)^i \mathrm{d}t}{\prod_{l=3}^{d} \Big(\tanh(t) - \frac{2x_l - b - a}{b-a}\Big)^{\frac{j}{N}} \cosh(t)^{2-\frac{2j}{N}}}$$

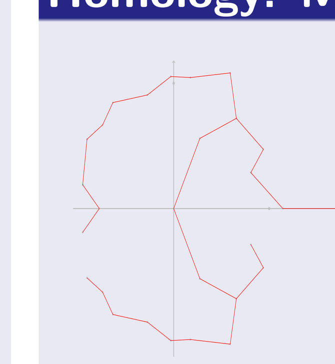
corresponds to the period of ω along a cycle $\alpha^{(0)} \in H_1(\mathcal{C},\mathbb{Z})$ starting on a fixed, but undetermined sheet $s^{(0)} \in \{1,\ldots,N\}$, encircling a and b (see picture to the right).



Using the transformation $y \mapsto \zeta^k y$ we obtain cycles $\alpha^{(k)} \in H_1(\mathcal{C},\mathbb{Z})$, similar to $\alpha^{(0)}$ except for starting on sheet $s^{(0)} + k \bmod N$. The corresponding periods are given by

$$\int_{\alpha^{(k)}} \omega = \zeta^{-jk} \int_{\alpha^{(0)}} \omega,\; k = 0, \ldots, N-1\,.$$

Homology: Maximal spanning tree



Construct a spanning tree w.r.t. maximal holomorphicity τ with edges $e_1, \ldots, e_{d-1}$ connecting the branch points. For an edge $e_i = [a,b]$ we compute the periods $\int_{\alpha_i^{(k)}} \omega$ with $\alpha_i^{(k)}$ corresponding to a and b as above.

In total, for each $\omega \in \mathcal{D}$ we obtain $(d-1)N > 2g$ periods, corresponding to the cycles $\alpha_i^{(k)}$.

The intersection matrix can be determined by analyzing the geometry of the spanning tree in the complex plane and has rank $2g$, therefore proving that the set $\{\, \alpha_i^{(k)} \,\}$ contains a basis for $H_1(\mathcal{C},\mathbb{Z})$.

Computing and applying a symplectic transformation yields the small period matrix $\mathcal{R}$.

References

- B. Deconinck and M. van Hoeij, Computing Riemann matrices of algebraic curves, Phys. D, 152/153:28–46, 2001.
- Pascal Molin, Intégration numérique et calculs de fonctions L, Thése de doctorat, Université de Bordeaux I, 2010