CARL VON OSSIETZKY universität OLDENBURG
FAKULTÄT V
MATHEMATIK UND NATURWISSENSCHAFTEN

Algorithms on Riemann surfaces - Construction of paths

IfM INSTITUT FÜR MATHEMATIK

Stefan Hellbusch - Carl von Ossietzky Universität Oldenburg

Setup and notation

Let $f \in \mathbb{Q}[x, y]$ be a geometrically irreducible polynomial of degree $\deg_y(f) = N$ in y and X the compact Riemann surface of genus g defined by f.

Denote by $y(x)$ an N-sheeted algebraic covering of $\mathbb{P}^1$, $\mathcal{B} = \{\, x \in \mathbb{P}^1(\mathbb{C}) \mid \#y(x) < N \,\}$ the (projected) branch points, and $\mathfrak{b} = \#\mathcal{B}$ the number of branch points.

The *Jacobian* of X is the complex torus $\mathrm{Jac}(X) := \mathbb{C}^g/\Lambda$, where Λ is the lattice generated by the columns of a so called (big) period matrix.

One goal and an algorithm

One goal is the fast and rigorous computation of a period matrix to high precision. This has many applications in number theory.

An algorithm for this is described in [1] and uses the following steps:

1 Compute $\mathcal{B}$ and a base point x_0.
2 Compute a generating set of the fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{B}, x_0)$.
3 Lift to the Riemann surface X and obtain the local monodromies.
4 Find a symplectic basis of the homology group $H_1(X, \mathbb{Z})$.
5 Find a basis of the holomorphic 1-forms $\Omega^1(X)$ on X.
6 Integrate the holomorphic 1-forms along the symplectic basis.
7 Obtain the period matrix.

If we are only interested in the local monodromies or the homology group, we can stop after the corresponding step.

For the computation of the period matrix, we have rearranged these steps. For more details see the poster of Christian Neurohr.

This poster focuses on step 2.

A theoretic method

To obtain a generating set of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{B}, x_0)$ one wants to create paths γ_i, starting and ending in x_0 and encircling each point of $x_i \in \mathcal{B}$ counterclockwise, such that no other point of $\mathcal{B}$ is encircled by the path.

These paths form a generating set of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{B}, x_0)$, but possibly one path goes from x_0 to a point at infinity, causing some problems in practice.

The Star-method

First, we followed the approach of [1]. Denote by $\mathcal{B}' \subset \mathbb{C}$ the subset of finite points of $\mathcal{B}$ and order them by their angle with the base point as center.

For this, we choose the base point to be in the middle of the points in $\mathcal{B}'$ in some sense and such that no two points of $\mathcal{B}'$ and the base point are colinear. This implies that by our ordering no two points are equal.

Applying the theoretic method above, we obtain a set of paths $\{\gamma_1, \dots, \gamma_{\mathfrak{b}'}\}$, with $\mathfrak{b}' = \#\mathcal{B}'$, which is a generating set of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{B}, x_0)$.

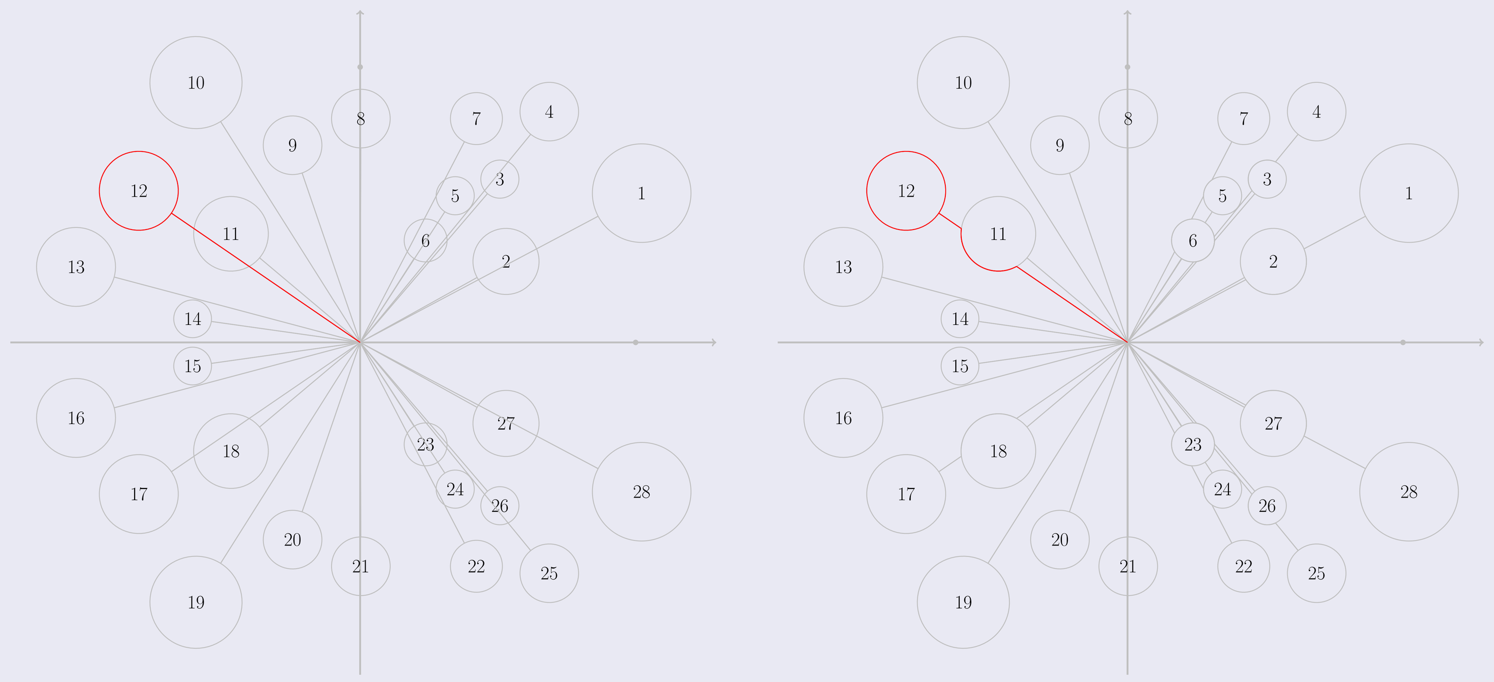
This follows because the possibly missing path around a point at infinity is homotopic to the inverse of the concatenation of the constructed paths:

$$\gamma_\infty \sim \left(\prod_{i=1}^{\mathfrak{b}'} \gamma_i\right)^{-1}$$

Images of the Star-method

The surface used for this picture is defined by the polynomial:

$$f = (x^2 + y)^5 + (x - y)^2 + 5x^3y^2 + 1 \in \mathbb{Q}[x, y]$$



In gray you see all paths $\gamma_1, \dots, \gamma_{28}$ around the 28 ordered points of $\mathcal{B}'$ and in red the path γ_{12} around point number 12.

For example, the arc at point number 11 in the right picture is added to avoid some numerical problems and the orientation is chosen so that the path is homotopic to the path in the left picture.

Spanning trees

Next, we implemented a minimal spanning tree (MST) method motivated by [2]. One problem is that with the old ordering, one needs to carefully choose the direction of the arcs to get a relation for the path γ_∞.

An easier way is a suitable reordering of the points in $\mathcal{B}'$. Let $K(\mathcal{B}' \cup \{x_0\})$ be the complete graph with vertices $\mathcal{B}'$ and T a spanning tree of this graph. Because T is a tree, there are unique paths from one vertex to another (where no edge is used more than once).

Applying a search algorithm induces an ordering of the vertices and respectively of $\mathcal{B}'$. We use a depth-first search and in the nondeterministic steps we use a criterion based on relative angles. This allows us to choose the direction of the arcs.

Overview of the MST-method

- Start with the complete graph with vertices $\mathcal{B}' \cup \{x_0\}$.
- Choose and apply a weight function $w : \mathcal{B}' \cup \{x_0\} \times \mathcal{B}' \cup \{x_0\} \to \mathbb{R}$.
- Compute a minimal spanning tree with respect to w.
- Find the paths and a suitable ordering using a depth-first strategy starting in the base point x_0.
- Each arc on the way to the current point is clockwise and the circle around the point remains counterclockwise.

This results in a generating set of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{B}, x_0)$ and the relation

$$\gamma_\infty \sim \left(\prod_{i=1}^{\mathfrak{b}'} \gamma_i\right)^{-1}$$

holds.

Examples for weight functions

- Euclidean distances: $w_e(x, y) = |x - y|$

This is motivated by the (unfortunately incorrect) assumption that short paths are simpler then longer ones. But for the local monodromies, [2] obtained an optimization using puiseux series, whose radius of convergence is the distance to the "closest" point in $\mathcal{B}$.

- Angle: $w_s(x, y)$. Consider the base point as the center, then move x and y to this center and take the argument. Then $w_s(x, y)$ is the minimum of those two values.

This results, more or less, in the Star-method (see the left image for the Star-method) and therefore our first method can be obtained by a spanning tree method.

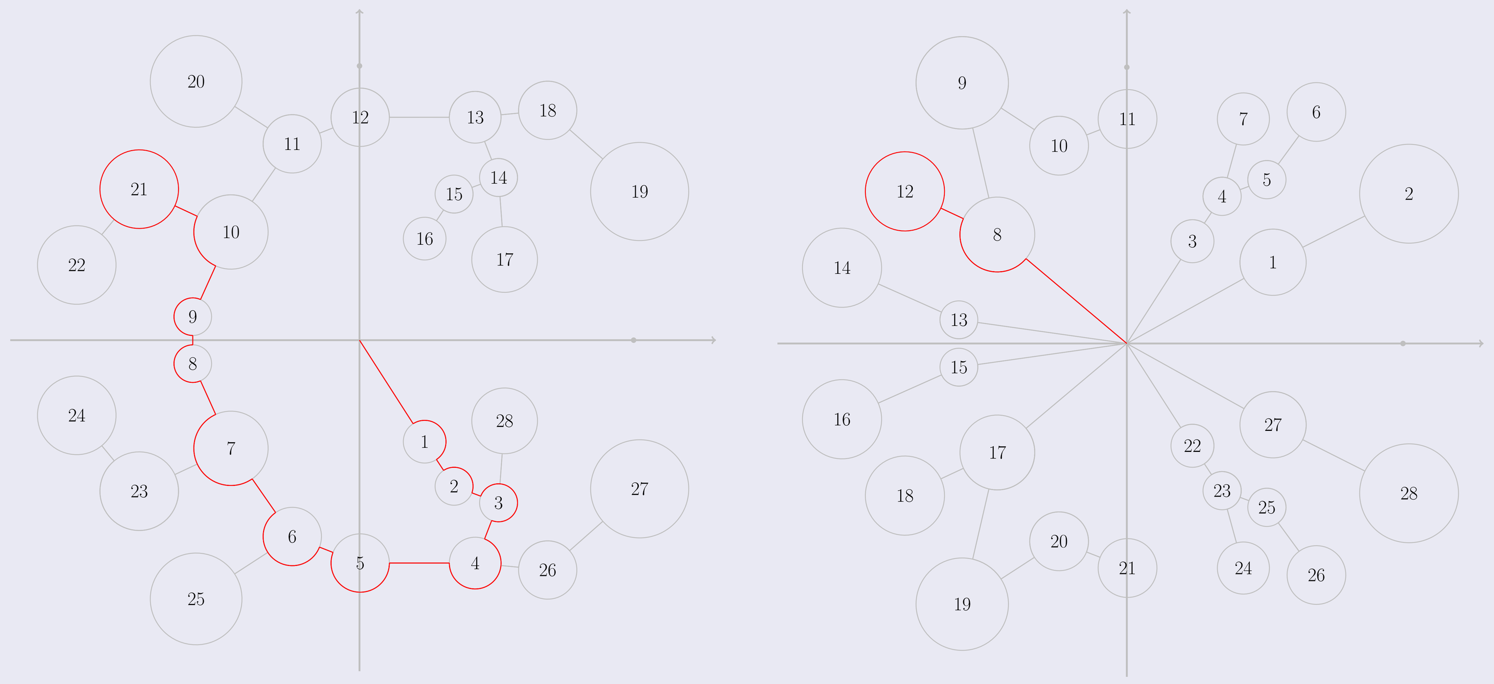
- Mixed: $w_m = c_1 w_1 + c_2 w_2$, with $w_{1,2}$ weight functions and $c_{1,2}$ constants.

With a careful choice of $c_{1,2}$, this function allow us to mix and compare two weight functions $w_{1,2}$.

- Holomorphic: $w_h(x, y) = -\tau(x, y)$. Here $\tau(x, y)$ is a parameter used for the integration of the straight line path from x to y. (See the poster of Christian Neurohr for more details.)

Images of MST-Methods

Coloring and polynomial as before.



On the left you see the minimal euclidean distances and on the right a mix of angle and distance.

Outlook (Work in progress)

Use puiseux series for continuation and integration.

The configuration of the points in $\mathcal{B}'$ are given by the defining polynomial. We are currently looking for a method to construct good transformations yielding better generating sets.

Shrink each circle to radius $r \to 0$ as in the superelliptic case seen on the poster of Christian Neurohr.

References

- B. Deconinck and M. van Hoeij, Computing Riemann matrices of algebraic curves, Phys. D, 152/153:28–46, 2001.
- A. Poteaux, Computing Monodromy Groups defined by Plane Algebraic Curves. In: Proceedings of the 2007 International Workshop on Symbolic-numeric Computation. ACM, New-York, 2007, 36–45