Arithmetical Properties of Hypergeometric Bernoulli Numbers

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Abstract

In a recent paper, Byrnes et al. have developed some recurrence relations for the hypergoemetric zeta functions. Moreover, the authors presented, in the same paper, two conjectures on arithmetical properties of the denominators of the reduced fraction of the hypergeometric Bernoulli numbers. In this poster, we point out our steps in proving of these conjectures using some recurrence relations. Furthermore, we observe that the above properties hold for both Carlitz and Howard numbers.

Introduction

The classical Bernoulli numbers B_k can be defined in a number of ways. One of the most common and useful methods is using the generating function

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

In 1961, Carlitz extended the above notion and introduced the coefficients, β_k as

$$\frac{x^2}{e^x-x-1}=\sum_{k=0}^{\infty}\beta_k\frac{x^k}{k!}$$

and stated that nothing is known about them. And more generally, in 1967, Howard defined the numbers $A_{k,r}$ by

$$\frac{x^k}{k!}\left(e^x-\sum_{n=0}^{k-1}\frac{x^n}{n!}\right)^{-1}=\sum_{r=0}^{\infty}A_{k,r}\frac{x^r}{r!}.$$

From this definition, by applying the Cauchy product formula for infinite series, one can obtain the following recurrence relation:

$$\sum_{r=0}^{n}\binom{n+k}{r}A_{k,r}=0, \qquad \text{for } n>0$$

with $A_{k,0}=1$.
In this poster, we consider hypergeometric Bernoulli numbers generated by the reciprocal of $\Phi_{1,b}(z)$ where $\Phi_{a,b}(z)$ is the Kummer's function defined by

$$\Phi_{a,b}(z) := {}_1F_1\left(\begin{matrix}a\\ a+b\end{matrix}\middle| z\right), \qquad \text{for } a,b\in\mathbb{R}.$$

Definition 1. *Let b be a natural number. The hypergeometric Bernoulli numbers $B_n(b)$ are defined by*

$$\frac{1}{\Phi_{1,b}(z)}=\sum_{n=0}^{\infty}B_n(b)\frac{z^n}{n!}.$$

These are precisely the numbers $A_{b,n}$ studied by Howard. One can observe that the special case $b=1$ corresponds to the Bernoulli numbers $B_n=B_n(1)$.

Introduction Contd...

The numbers $B_n(b)$ are also expressed in terms of the hypergeometric zeta function $\zeta^H_{1,b}(s)$ as

$$B_n(b)=\begin{cases}1 & \text{for } n=0,\\ -1/(1+b) & \text{for } n=1,\\ -n!\zeta^H_{1,b}(n)/b & \text{for } n\geq 2.\end{cases}$$

Where the hypergeometric zeta function $\zeta^H_{1,b}(s)$ is given by

$$\zeta^H_{1,b}(s):=\sum_{k=1}^{\infty}\frac{1}{z^s_{k;1,b}}, \qquad \Re(s)>1$$

with $z_{k;1,b}$ is the sequence of complex zeros of the function $\Phi_{1,b}(z)$.

Preliminaries and Motivations

It is well known that the Bernoulli numbers of odd indices vanish except $B_1=-1/2$, leaving $B_{2n}=N_{2n}/D_{2n}$ in reduced form. One of the basic arithmetic properties of D_{2n} is the following:

Theorem 1 (von Staudt-Clausen Theorem). *The Denominator of the Bernoulli number B_{2n} is given by*

$$D_{2n}=\prod_{(p-1)|2n}p.$$

From this theorem, one can easily see that the denominator of B_{2n} is divisible by 6 and square free. In the case of hypergeometric Bernoulli numbers, as indicated in [1] computer experiment suggest an extension of these properties.
Let $\mathfrak{D}(b)=$ (denominator of the reduced fraction $(B_n(b))$: $n\geq 0$). Then some of the examples given in [1] were

$$\begin{aligned}\mathfrak{D}(2) &= (1,3,18,90,270,1134,5670,2430,\ldots),\\ \mathfrak{D}(3) &= (1,4,40,160,5600,896,19200,76800,\ldots),\\ \mathfrak{D}(4) &= (1,5,75,875,26250,78750,918750,\ldots).\end{aligned}$$

Based on data like these, it was conjectured in [1] that

Conjecture 1. *Every prime p dividing the the denominator of the reduced fraction for $B_n(b)$ satisfies $p\leq n+b$.*

Preliminaries and Motivations Contd...

Conjecture 2. *Let $\alpha(b)$ be the number of odd terms at the beginning of $\mathfrak{D}(b)$. Then*

$$\alpha(b)=2^{v_2(b)}$$

where $v_2(b)$ is the highest power of 2 that divides b.

According to [1] these two conjectures have been verified up to $b=1000$.

Main Results

Lemma 1. *Let $a,b:\mathbb{N}\to\mathbb{C}$, $a(0)=b(0)=1$. Then*

$$\left(\sum_{k=0}^{\infty}a(k)q^k\right)\left(\sum_{k=0}^{\infty}b(k)q^k\right)=1$$

if and only if

$$a(k)=-\sum_{n=1}^{k}a(k-n)b(n), \qquad \text{for } k\geq 1.$$

By using the above lemma we can obtain

$$B_n(b)=-\sum_{k=1}^{n}\frac{n(n-1)\cdots(n-k+1)}{(1+b)_k}B_{n-k}(b), \qquad \text{for } n\geq 1.$$

Proposition 1. *Let $\alpha(b)$ be the number of odd terms at the beginning of $\mathfrak{D}(b)$. Then for $b\not\equiv 0 \pmod 4$, we have $\alpha(b)=v_2(b)+1$.*

Lemma 2. *Let $m,n\in\mathbb{N}$ and let*

$$P_{n,m}(b):=\prod_{k=0}^{m}\frac{n-k}{1+k+b}$$

with $m<n\leq 2^{v_2(b)}-1$ and b a multiple of four. Then the denominator of the reduced fraction for $P_{n,m}(b)$ is an odd integer.

Theorem 2. *Let $\alpha(b)$ be the number of odd terms at the beginning of $\mathfrak{D}(b)$. Then the function α is completely multiplicative, that is*

$$\alpha(b)=2^{v_2(b)}.$$

Theorem 3. *Let $v_p(n,b)$ be the largest prime that divides the denominator of the reduced fraction for $B_n(b)$. Then $v_p(n,b)\leq n+b$.*

It is easy to see that the Carlitz coefficients β_n can be expressed as $\beta_n=2B_n(2)$. The numbers β_n are rational and satisfy the following property: a prime p which divides the denominator of the reduced fraction for β_n is at most $n+2$. Similarly, if a prime p divides the denominator of the reduced fraction for Howard numbers $A_{b,n}$, then $p\leq n+b$.

Reference. [1] *A. Byrnes, L. Jiu, V. H. Moll, and C. Vignat. Recursion rules for the hypergeometric zeta function. International Journal of Number Theory, 10(07):1761-1782, 2014.*