Elliptic factors in Jacobians of hyperelliptic curves with certain automorphism groups

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My original interest in Jacobian variety decomposition was motivated by the following question.

Question

Given a genus g, what is the largest integer t such that there is some curve X of genus g with $J_X \sim E^t \times A$ for some elliptic curve E and an abelian variety A?

The $\dim(J_X) = g$, so the largest t can possibly be is g.

Non-hyperelliptic Curves

Genus	Auto. Group	Jacobian Decomposition
4	$(72, 40)$	$J_X \sim E^4$
5	$(160, 234)$	$J_X \sim E^5$
6	$(72, 15)$	$J_X \sim E^6$
7	$\mathrm{PSL}(2, 7)$	$J_X \sim E^7$
8	$(336, 208)$	$J_X \sim E^8$
9	$(192, 955)$	$J_X \sim E_1^3 \times E_2^6$
10	$(360, 118)$	$J_X \sim E^{10}$
14	$\mathrm{PSL}(2, 13)$	$J_X \sim E^{14}$

Decomposition Techniques

- X a curve of genus g
- J_X its Jacobian Variety
- G the automorphism group of X

The techniques work for curves defined over any field. But a field must be specified to compute the automorphism group of the curve.

We assume all curves are defined over an algebraically closed field of characteristic zero.

From a theorem of Wedderburn we know that

$$\mathbb{Q}[G] \cong \bigoplus_i M_{n_i}(\Delta_i)$$

where Δ_i are division rings.

$\pi_{i,j} \in \mathbb{Q}[G]$ with the zero matrix in every component except the ith component where it has a 1 in the j,j position and zeros elsewhere.

Apply the natural map of $\mathbb{Q}$-algebras $e : \mathbb{Q}[G] \to \mathrm{End}(J_X) \otimes_{\mathbb{Z}} \mathbb{Q}$ a result of Kani-Rosen:

$$J_X \sim \bigoplus_{i,j} e(\pi_{i,j}) J_X.$$

What are these $e(\pi_{i,j})J_X$? **Recall**: We want to find elliptic curve factors.

For a special $\mathbb{Q}$-character χ

$$\dim e(\pi_{i,j})J_X = \frac{1}{2}\langle \chi, \chi_i \rangle$$

where the χ_i are the irreducible $\mathbb{Q}$-characters.

Take the quotient map from X to $Y = X/G$, branched at s points with monodromy $g_1, \dots, g_s \in G$.

$$g_1 \cdot g_2 \cdots g_s = 1_G \quad \text{and} \quad \langle g_1, g_2, \dots, g_s \rangle = G$$

Definition

A **Hurwitz character** of a group G is a character of the form:

$$\chi = 2\chi_{\text{triv}} + 2(g_Y - 1)\,\chi_{\langle 1_G \rangle} + \sum_{i=1}^{s} \left(\chi_{\langle 1_G \rangle} - \chi_{\langle g_i \rangle} \right)$$

$\chi_{\langle g_i \rangle}$ is the character of G induced from the trivial character of $\langle g_i \rangle$ and χ_{triv} is the trivial character of G.

(**Remember later**: two elements in the same conjugacy class generate the same induced character.)

map from X to $Y = X/G$, branched at s points with monodromy $g_1, \ldots, g_s \in G$

$$\chi = 2\chi_{\text{triv}} + 2\,(g_Y - 1)\,\chi_{\langle 1_G \rangle} + \sum_{i=1}^{s} \left(\chi_{\langle 1_G \rangle} - \chi_{\langle g_i \rangle}\right)$$

To compute a Hurwitz character we need to know:

- signature – $(m_1, m_2, \ldots, m_s)$ where m_i is order of g_i.
- monodromy – $g_1, \ldots, g_s$

To compute the dimension of the factor, we also need

- the irreducible $\mathbb{Q}$-characters

$$\mathbb{Q}[G] \cong \bigoplus_i M_{n_i}(\Delta_i) \quad \text{and} \quad J_X \sim \bigoplus_{i,j} e(\pi_{i,j}) J_X$$

Recall: We want to find lots of isogenous elliptic curves.

Theorem (P., '07)

With notation as above, $e(\pi_{i,j})J_X$ is isogenous to $e(\pi_{i,k})J_X$.

If there is some i with $\frac{1}{2}\langle \chi, \chi_i \rangle = 1$, then there are n_i isogenous elliptic curves in the factorization of J_X.

Low Genus Results

Brandt and Stichtenoth ('86) and Shaska ('03) completely classify all automorphism groups of hyperelliptic curves of any genus over an algebraically closed field of characteristic zero.

ω the hyperelliptic involution, then the reduced automorphism group ($G/\langle\omega\rangle$) must be a dihedral group, a cyclic group, A_4, S_4, or A_5.

We consider hyperelliptic curves with reduced automorphism group one of A_4, S_4, or A_5.

- signature
- monodromy
- irreducible $\mathbb{Q}$-characters

Signatures are in Shaska's paper.

Recall: two elements in the same conjugacy class will generate the same induced character. For small cases we can search through the group to find elements of the group satisfying the monodromy conditions

Character tables for these groups are well known.

Theorem (P.)

The hyperelliptic curve of genus 4 with affine model

$$X : y^2 = x(x^4 - 1)(x^4 + 2\sqrt{-3}\, x^2 + 1)$$

has a Jacobian variety that decomposes as $E_1^2 \times E_2^2$ for two elliptic curves E_i.

Theorem (P.)

The genus 5 hyperelliptic curve with affine model

$$X : y^2 = x(x^{10} + 11x^5 - 1)$$

has $J_X \sim E^5$ for the elliptic curve $E : y^2 = x(x^2 + 11x - 1)$.

Genus	Auto. Group	Dim.	Jac. Decomp.
4	$\mathrm{SL}_2(3)$	0	$E_1^2 \times E_2^2$
5	$A_4 \times C_2$	1	$E^3 \times A_2$
	$W_2 = (48, 30)$	0	$E_1^2 \times E_2^3$
	$A_5 \times C_2$	0	E^5
6	$\mathrm{GL}_2(3)$	0	$E_1^2 \times E_2^4$
7	$A_4 \times C_2$	1	$E_1 \times E_2^3 \times E_3^3$
8	$\mathrm{SL}_2(3)$	1	$A_{2,1}^2 \times A_{2,2}^2$
	$W_3 = (48, 28)$	0	$E^4 \times A_2^2$
9	$A_4 \times C_2$	1	$E^3 \times A_2^3$
	W_2	0	$E_1 \times E_2^2 \times A_2^3$
	$A_5 \times C_2$	0	$E_1^4 \times E_2^5$
10	$\mathrm{SL}_2(3)$	1	$A_2^2 \times A_3^2$

Help From a Computer Program

Thomas Breuer wrote a program which classifies all automorphism groups of Riemann Surfaces for a given genus g. It requires a search through a database of all groups up to a certain order.

In the late 1990s he ran it in GAP3 for genus up to 48. GAP had complete classification of groups up to order 1000 (except 512 and 768).

Branching data is computed in the execution of his algorithms but was not recorded.

I rewrote the program in MAGMA and added functionality to output monodromy data.

input the known signature and known automorphism group of a curve and output elements of the monodromy

Breuer devised a (recursive) algorithm to handle higher genus but never implemented it. I have also started to implement the higher genus algorithm.

Genus	Automorp. Group	Dimen.	Jacobian Decomposition
11	$A_4 \times C_2$	2	$A_2 \times A_3^3$
	$S_4 \times C_2$	1	$E^3 \times A_{2,1} \times A_{2,2}^3$
12	$\mathrm{SL}_2(3)$	1	$A_2^2 \times A_4^2$
	W_3	0	$A_{2,1}^2 \times A_{2,2}^4$
13	$A_4 \times C_2$	2	$E \times A_{3,1} \times A_{3,2}^3$
14	$\mathrm{SL}_2(3)$	2	$A_3^2 \times A_4^2$
	$\mathrm{GL}_2(3)$	1	$A_2^4 \times A_3^2$
	$\mathrm{SL}_2(5)$	0	$E_1^4 \times E_2^6 \times A_2^2$
15	$A_4 \times C_2$	2	$A_2^3 \times A_3^3$
	$S_4 \times C_2$	1	$E \times E_2^2 \times A_4^3$
	$A_5 \times C_2$	0	$E_1^4 \times E_2^5 \times A_2^3$

Genus	Automorp. Group	Dimen.	Jacobian Decomposition
16	$\mathrm{SL}_2(3)$	2	$A_3^2 \times A_5^2$
17	$A_4 \times C_2$	3	$E \times A_{4,1} \times A_{4,2}^3$
	W_2	1	$E \times A_2^2 \times A_4^3$
18	$\mathrm{SL}_2(3)$	2	$A_3^2 \times A_6^2$
	$\mathrm{GL}_2(3)$	1	$A_{3,1}^2 \times A_{3,2}^4$
19	$A_4 \times C_2$	3	$E \times A_2^3 \times A_4^3$
20	$\mathrm{SL}_2(3)$	3	$A_4^2 \times A_6^2$
	W_3	1	$A_{2,1}^2 \times A_{2,2}^2 \times A_3^4$
	$\mathrm{SL}_2(5)$	0	$E^4 \times A_{2,1}^2 \times A_{2,2}^6$

The End