Fast computation of isomorphisms of hyperelliptic curves and explicit descent

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Motivation in genus 1

Let K be an algebraically closed field of characteristic $p \neq 2$.

- Elliptic curves $(p \neq 3)$ $E/K \ : y^2 = x^3 + a\,x + b$ are classified up to isomorphism by

$$j(E) = 1728\,\frac{4a^3}{4\,a^3 + 27\,b^2}\,.$$

- Conversely, for any $j \in K \setminus \{0, 1728\}$, we can reconstruct a curve E s.t. $j(E) = j$, for instance

$$E/K : y^2 = x^3 - \frac{27\,j}{j - 1728}\,x + \frac{54\,j}{j - 1728}\,.$$

General genus

- Similarly, we would like to do the same for hyperelliptic curves of genus $g \geq 2$, *i.e.* $C/K : y^2 = f(x)$ with $\deg(f) = 2g + 2$ and simple roots.

$$\{\text{Hyperelliptic curves of genus } g\}_{/\simeq} \longleftrightarrow \{\text{a `space' of parameters}\}$$

- More precisely, given two such curves represented by the same parameters, we would like to find an explicit isomorphism between them.

Applications

- Determining automorphism groups of curves;
- Galois descent for curves;
- Geometric and arithmetic information on the moduli space;
- Reconstructing curves from invariants;
- Applications to cryptography (CM method).

Isomorphisms

Let $C : y^2 = f(x)$ and $C' : y^2 = f'(x)$ be two hyperelliptic curves of genus g. Every isomorphism from C to C' is of the form

$$(x, y) \mapsto \left(\frac{ax + b}{cx + d}, \frac{ey}{(cx + d)^{g+1}} \right)$$

for some $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(K)$ and $e \in K^*$.

Isomorphisms

Let $C : y^2 = f(x,z)$ and $C' : y^2 = f'(x,z)$ be two hyperelliptic curves of genus g in weighted projective $(1,1,g+1)$-space. Every isomorphism from C to C' is of the form

$$(x,z,y) \mapsto (ax+bz, cx+dz, ey)$$

for some $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(K)$ and $e \in K^*$.

Invariants

Definition

Let $M^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(K)$ act on binary forms $f(x,z)$ of even degree n by $M.f = f(ax+bz, cx+dz)$.

A homogenous polynomial function I on the space of such forms f is an invariant if there exists $\omega \in \mathbb{Z}$ such that for all $M \in \mathrm{GL}_2(K)$,

$$I(M.f) = \det(M)^{\omega} \cdot I(f).$$

Let n, resp. d, be the degree of f, resp. I. If nd is odd then I is zero. Otherwise we have the equality $\omega = nd/2$ for the weight ω of C.

Ex: $f = a_2X^2 + a_1XZ + a_0Z^2$, $I = a_1^2 - 4a_2a_0$ is a degree-2 invariant.

Invariants and isomorphisms

Fact: the algebra of invariants $\mathcal{I}_n$ is finitely generated (Gordan 1868) and for $n \leq 10$ generators are explicitly known.

Theorem (- Mumford 1977)

Let f, f' be binary forms of even degree $n \geq 4$ with simple roots. Let $\{I_i\}$ be a finite set of homogeneous generators of degree d_i for $\mathcal{I}_n$.

Then f and f' are in the same orbit under the action of $\mathrm{GL}_2(K)$ if and only if there exists $\lambda \in K$ such that for all i, $I_i(f) = \lambda^{d_i} \cdot I_i(f')$.

So we can test efficiently whether $C : y^2 = f(x)$ and $C' : y^2 = f'(x)$ are isomorphic by computing a finite set of invariants. But how to obtain these?

Covariant and transvectant

To construct invariants, one needs to embed them in a broader framework.

Definition

A homogeneous polynomial function $C : f \mapsto g$ sending binary forms f of degree n to binary forms g of degree r is a covariant if for all $M \in \mathrm{GL}_2(K)$,

$$C(M.f) = \det(M)^{\omega} \cdot M.C(f).$$

The integer r is called the order of C. If $nd - r$ is odd, C is zero. Otherwise we have the equality $\omega = (nd - r)/2$ for the weight ω of C.

Ex: The identity map is a covariant of order n, degree 1 and weight 0.

We will identify C with $C(f)$ for the tautological form $f \in F(a_0, \ldots, a_n)[x, z]$. Here F is the prime field of K.

On the algebra $\mathcal{C}_n$ of covariants, there are bilinear differential operators, called *h*-th transvectant

$$\Big(\underbrace{C_1}_{\substack{\text{degree } d_1 \\ \text{order } r_1}} , \underbrace{C_2}_{\substack{\text{degree } d_2 \\ \text{order } r_2}} \Big) \mapsto \underbrace{(C_1, C_2)_h}_{\substack{\text{degree } d_1 + d_2 \\ \text{order } r_1 + r_2 - 2h}}$$

Fact (Gordan 1868): starting from the covariant f and applying a finite number of h-th transvectants, one can get a set of generators for $\mathcal{I}_n$ (and for $\mathcal{C}_n$).

Genus 1

Let

$$f = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

There is one covariant of degree 2 and order 4

$$(f,f)_2 = (1/3a_2a_4 - 1/8a_3^2)x^4 + (a_1a_4 - 1/6a_2a_3)x^3 + (2a_0a_4 + 1/4a_1a_3 - 1/6a_2^2)x^2 + (a_0a_3 - 1/6a_1a_2)x + 1/3a_0a_2 - 1/8a_1^2.$$

The algebra of invariants $\mathcal{I}_4$ is generated by

$$I = (f,f)_4 = 2a_0a_4 - 1/2a_1a_3 + 1/6a_2^2$$

and by

$$J = (f,(f,f)_2)_4 = a_0a_2a_4 - 3/8a_0a_3^2 - 3/8a_1^2a_4 + 1/8a_1a_2a_3 - 1/36a_2^3.$$

Rem: The j-invariant is equal to $1728I^3/(I^3 - 6J^2)$.

Computing isomorphisms

Proposition

Let $C_i : y^2 = f_i(x)$ be hyperelliptic curves of genus g. Let c_i be covariants of f_i with non-zero discriminant and $X_i : y^2 = c_i(x)$ the associated hyperelliptic curves. Then, up to the hyperelliptic involution, $\mathcal{I}som(C_1, C_2) \subset \mathcal{I}som(X_1, X_2)$.

Hence, one can recursively reduce the computation to lower genera and/or use a new basic method to deal with this easier case.

Generically, one can use the quartic covariant $(f, f)_{n-2}$. This yields fast algorithms:

Field	Method	Genus g										
		1	2	4	8	16	32	64	128	256	512	1024
$\mathbb{F}_{10007}$	`IsGL2Equivalent`	0	0	0	0	0.1	0.2	0.9	6.5	39	242	1560
	`IsGL2EquivFast`	0	0	0	0	0	0	0.1	0.6	3.7	25	165
	`IsGL2EquivCovariant`	0	0	0	0	0	0	0	0	0.1	0.5	2.5
$\mathbb{Q}$	`IsGL2Equivalent`	0	0	0.4	15	1150	-	-	-	-	-	-
	`IsGL2EquivFast`	0	0	0	0	0.1	0.2	0.6	3	30	382	5850
	`IsGL2EquivCovariant`	0	0	0	0	0	0	0	0.2	0.6	3.4	7

Galois descent

So far, we worked over an algebraically closed field, but what happens if now $k \subset \overline{k} = K$ is any field (of characteristic 0 or a finite field) ?

Definition

Let C/K be a curve of genus $g \geq 2$.

A field k is a field of definition for C if there exists a curve $\mathcal{C}/k$ (called a model of C) which is K-isomorphic to C.

The intersection $\mathbf{M}_C$ of all the fields of definition is called the field of moduli of C.

One has also $\mathbf{M}_C = K^H$ where $H = \{\sigma \in \mathrm{Aut}(K),\ C \simeq {}^{\sigma}C\}$ and it is the residue field of the point $[C]$ in the coarse moduli space M_g.

$\mathbf{M}_C$ is a field of definition when

- C has no automorphisms;
- K is the algebraic closure of a finite field.

Galois descent and covariants

Theorem

Let $C : y^2 = f(x)$, let c be a covariant of f with non-zero discriminant and let $X : y^2 = c(x)$ be the associated curve. Suppose that X is (hyperelliptically) defined over its field of moduli.
Then C is (hyperelliptically) defined over an extension of its field of moduli of degree at most $[\mathrm{Aut}_K(X) : \# \mathrm{Aut}_K(C)]$.

The proof yields the following explicit descent method:

- Calculate a non-degenerate covariant c of f;
- Descend the covariant curve X (automatic in genus 1);
- Compute the descent morphism by our earlier algorithms;
- Apply the descent morphism to C.

Example for $g = 3$ with C_2^3

$$(j_2 : j_3 : \dots : j_{10}) = \left(0 : 0 : -\frac{25}{98} : -\frac{25}{98} : -\frac{225}{2744} : -\frac{25}{1372} : -\frac{225}{134456} : \frac{1125}{76832} : \frac{15125}{3764768}\right).$$

This gives rise to the curve $C : y^2 = f(x)$ with $\mathrm{Aut}_K(C) \simeq C_2^3$ and

$$f(x) = (-32\,\alpha^2 + 420\,\alpha - 2275)/160\,x^8 + (-12\,\alpha^2 + 140\,\alpha - 700)/25\,x^6 + \alpha\,x^4 + x^2 + (16\,\alpha^2 + 280\,\alpha - 2275)/12250$$

over $\mathbb{Q}(\alpha)$, where $\alpha^3 - 35/2\,\alpha^2 + 1925/16\,\alpha - 18375/64 = 0$.

Take the covariant curve $X : y^2 = c(x)$ with $\mathrm{Aut}_K(X) \simeq C_2^3$ where

$$c = (f,f)_6 = (-16\,\alpha^2 + 180\,\alpha - 875)/280\,x^4 + (24\,\alpha^2 - 630\,\alpha + 3150)/1225\,x^2 + (4\,\alpha + 35)/490.$$

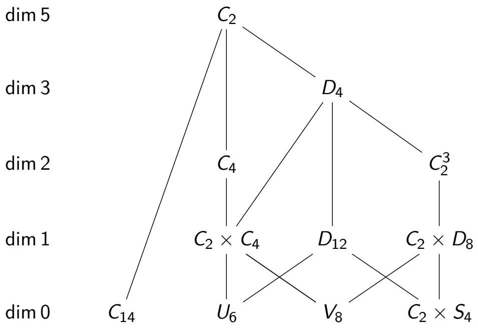
$I = -75/49$, $J = -2025/343$ so $X \simeq_K \mathfrak{X} : y^2 = x^3 + 25/9\,x + 25/9$.

We compute $\phi : X \to \mathfrak{X}$ and apply it to C:

$$\phi(C) : y^2 = x^8 + 160\,x^7 - 560\,x^6 - 2800\,x^5 + 64750\,x^4 - 91000\,x^3 + 3010000\,x^2 - 2225000\,x - 9696875\,.$$

Reconstruction in genus 3

$g = 3$ (char $K \neq 2, 3, 5, 7$)



- Reconstruction is possible for the C_2 and C_4 cases by Mestre's method;
- Gröbner basis methods give results for the strata of dimension ≤ 1;
- For C_2^3, these methods yield an extension, but we can descend as before;
- For D_4, a descent to the field of moduli does not always exist.

The D_4 case and beyond genus 3

The *reduced automorphism group* $\overline{\mathrm{Aut}}(C)$ of C is $\mathrm{Aut}(C)$ modulo the hyperelliptic involution.

Theorem (Huggins 2007)

Let C/K be a hyperelliptic curve whose reduced automorphism group is not cyclic. Then its field of moduli is a field of definition.

For general g and $|\overline{\mathrm{Aut}}(C)|$, work in progress has made explicit the obstruction for C to be defined over its field of moduli. It is determined by the splitting of a certain quaternion algebra determined by the invariants of C.

Conclusion

- For $g = 3$, extend our results to small characteristics $2 \leq p \leq 7$ (Lercier - Basson).
- For hyperelliptic curves, prove that if $p > 2g + 1$, Gordan's method generates all invariants.
- For hyperelliptic curves, develop our functions in Sage (work in progress by Rovetta).
- For hyperelliptic curves, develop algorithms to compute twists over finite fields (work in progress by Rovetta).
- Generalize the computations of isomorphisms to ternary forms (work in progress for plane quartics; *cf.* earlier results by Van Rijnswou).