ITERATED COLEMAN INTEGRATION FOR HYPERELLIPTIC CURVES

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Abstract. The Coleman integral is a p-adic line integral. Double Coleman integrals on elliptic curves appear in Kim's nonabelian Chabauty method, the first numerical examples of which were given by the author, Kedlaya, and Kim [3]. This paper describes the algorithms used to produce those examples, as well as techniques to compute higher iterated integrals on hyperelliptic curves, building on previous joint work with Bradshaw and Kedlaya [2].

1. Introduction

In a series of papers in the 1980s, Coleman gave a p-adic theory of integration on the projective line [6], then on curves and abelian varieties [7, 8]. This integration theory relies on locally defined antiderivatives that are extended analytically by the principle of Frobenius equivariance. In joint work with Bradshaw and Kedlaya [2], we made this construction explicit and gave algorithms to compute single Coleman integrals for hyperelliptic curves.

Having algorithms to compute Coleman integrals allows one to compute p-adic regulators in K-theory [6, 8], carry out the method of Chabauty-Coleman for finding rational points on higher genus curves [11], and utilize Kim's nonabelian analogue of the Chabauty method [10].

Kim's method, in the case of rank 1 elliptic curves, allows one to find integral points via the computation of double Coleman integrals. Indeed, Coleman's theory of integration is not limited to single integrals; it gives rise to an entire class of locally analytic functions, the *Coleman functions*, on which antidifferentiation is well-defined. In other words, one can define iterated p-adic integrals [4, 6]

$$\int_P^Q \xi_n \cdots \xi_1$$

which behave formally like iterated path integrals

$$\int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_n) \cdots f_1(t_1)\, dt_n \cdots dt_1.$$

Let us fix some notation. Let C be a genus g hyperelliptic curve over an unramified extension K of $\mathbb{Q}_p$ having good reduction. Let $k = \mathbb{F}_q$ denote its residue field, where $q = p^m$. We will assume that C is given by a model of the form $y^2 = f(x)$, where f is a monic separable polynomial with $\deg f = 2g + 1$.

Our methods for computing iterated integrals are similar in spirit to those detailed in [2]. We begin with algorithms for tiny iterated integrals, use Frobenius equivariance to write down a linear system yielding the values of integrals between points in different residue disks, and, if needed, use basic properties of integration to correct endpoints. We begin with some basic properties of iterated path integrals.

2. Iterated path integrals

We follow the convention of Kim [10] and define our integrals as follows:

$$\int_P^Q \xi_1\xi_2\cdots\xi_{n-1}\xi_n := \int_P^Q \xi_1(R_1)\int_P^{R_1}\xi_2(R_2)\cdots\int_P^{R_{n-2}}\xi_{n-1}(R_{n-1})\int_P^{R_{n-1}}\xi_n,$$

for a collection of dummy parameters $R_1,\ldots,R_{n-1}$ and 1-forms $\xi_1,\ldots,\xi_n$.

We begin by recalling some key formal properties satisfied by iterated path integrals [5].

Proposition 2.1. *Let $\xi_1,\ldots,\xi_n$ be 1-forms, holomorphic at points P,Q on C. Then the following are true:*

(1) $\int_P^P \xi_1\xi_2\cdots\xi_n = 0$,
(2) $\sum_{\text{all permutations }\sigma}\int_P^Q \omega_{\sigma(i_1)}\omega_{\sigma(i_2)}\cdots\omega_{\sigma(i_n)} = \prod_{j=1}^n \int_P^Q \omega_{i_j}$,
(3) $\int_P^Q \omega_{i_1}\cdots\omega_{i_n} = (-1)^n\int_Q^P \omega_{i_n}\cdots\omega_{i_1}$.

As an easy corollary of Proposition 2.1(2), we have

Corollary 2.2. *For a 1-form ω_i and points P,Q as before,*

$$\int_P^Q \omega_i\omega_i\cdots\omega_i = \frac{1}{n!}\left(\int_P^Q \omega_i\right)^n.$$

When possible, we will use this to write an iterated integral in terms of a single integral.

3. p-adic cohomology

We briefly recall some p-adic cohomology from [9], necessary for formulating the integration algorithms.

Let C' be the affine curve obtained by deleting the Weierstrass points from C, and let $A = K[x,y,z]/(y^2 - f(x), yz-1)$ be the coordinate ring of C'. Let $A^\dagger$ denote the Monsky-Washnitzer weak completion of A; it is the ring consisting of infinite sums of the form

$$\left\{\sum_{i=-\infty}^{\infty}\frac{B_i(x)}{y^i},\ B_i(x)\in K[x], \deg B_i \le 2g\right\},$$

further subject to the condition that $v_p(B_i(x))$ grows faster than a linear function of i as $i\to\pm\infty$. We make a ring out of these using the relation $y^2 = f(x)$.

These functions are holomorphic on the space over which we integrate, so we consider odd 1-forms written as

$$\omega = g(x,y)\frac{dx}{2y},\quad g(x,y)\in A^\dagger.$$

Any such differential can be written as

$$\omega = df + c_0\omega_0 + \cdots + c_{2g-1}\omega_{2g-1}, \tag{3.1}$$

with $f\in A^\dagger, c_i\in K$, and

$$\omega_i = x^i\,\frac{dx}{2y}\quad (i=0,\ldots,2g-1).$$

Namely, the set of differentials $\{\omega_i\}_{i=0}^{2g-1}$ forms a basis of the odd part of the de Rham cohomology of $A^\dagger$, which we denote as $H^1_{dR}(C')^-$.

To compute the p-power Frobenius action ϕ^* on $H^1_{dR}(C')^-$, one does the following:

- Let ϕ_K denote the unique automorphism lifting Frobenius from $\mathbb{F}_q$ to K. Extend ϕ_K to $A^\dagger$ by setting
$$\begin{aligned}\phi(x) &= x^p \\ \phi(y) &= y^p \left(1 + \frac{\phi(f)(x^p) - f(x)^p}{f(x)^p}\right)^{1/2} \\ &= y^p \sum_{i=0}^{\infty} \binom{1/2}{i} \frac{(\phi(f)(x^p) - f(x)^p)^i}{y^{2pi}},\end{aligned}$$
and
- use the relations
$$\begin{aligned}y^2 &= f(x) \\ d(x^i y^j) &= (2ix^{i-1}y^{j+1} + jx^i f'(x) y^{j-1})\frac{dx}{2y}\end{aligned}$$
to reduce large powers of x and large (in absolute value) powers of y to write $\phi^*(\omega)$ in the form (3.1).

This reduction process is known as *Kedlaya's algorithm* [9], and we will repeatedly use this algorithm to reduce iterated integrals involving $\omega \in A^\dagger \frac{dx}{2y}$ to iterated integrals in terms of basis elements ω_i.

4. Integrals: lemmas

Recall that we use Kedlaya's algorithm to compute single Coleman integrals as follows:

Algorithm 4.1 (Coleman integration in non-Weierstrass disks [2])**.**
Input: The basis differentials $(\omega_i)_{i=0}^{2g-1}$, points $P, Q \in C(\mathbb{C}_p)$ in non-Weierstrass residue disks, and a positive integer m such that the residue fields of P, Q are contained in $\mathbb{F}_{p^m}$.
Output: The integrals $\left(\int_P^Q \omega_i\right)_{i=0}^{2g-1}$.

(1) Calculate the action of the m-th power of Frobenius on each basis element (see Remark 4.2):
$$(\phi^m)^* \omega_i = dh_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j.$$
(2) By change of variables, we obtain
$$\sum_{j=0}^{2g-1} (M - I)_{ij} \int_P^Q \omega_j = h_i(P) - h_i(Q) - \int_P^{\phi^m(P)} \omega_i - \int_{\phi^m(Q)}^Q \omega_i \tag{4.1}$$
(the *fundamental linear system*). Since the eigenvalues of the matrix M are algebraic integers of $\mathbb{C}$-norm $p^{m/2} \neq 1$ (see [9, §2]), the matrix $M - I$ is invertible, and we may solve (4.1) to obtain the integrals $\int_P^Q \omega_i$.

Remark 4.2. To compute the action of ϕ^m, first carry out Kedlaya's algorithm to write

$$\phi^*\omega_i = dg_i + \sum_{j=0}^{2g-1} B_{ij}\omega_j.$$

If we view h, g as column vectors and M, B as matrices, induction on m shows that

$$\begin{aligned} h &= \phi^{m-1}(g) + B\phi^{m-2}(g) + \cdots + B\phi_K(B)\cdots\phi_K^{m-2}(B)g \\ M &= B\phi_K(B)\cdots\phi_K^{m-1}(B). \end{aligned}$$

Note, however, that when points $P, Q \in C(\mathbb{C}_p)$ are in the same residue disk, the "tiny" Coleman integral between them can be computed using a local parametrization, just as in the case of a real-valued line integral. This is also true when the integrals are iterated (see Section 5).

However, to compute general iterated integrals, we will need to employ the analogue of "additivity in endpoints" to link integrals between different residue disks. First, let us consider the case where we are breaking up the path by one point.

Lemma 4.3. *Let P, P', Q be points on C such that a path is to be taken from P to Q via P'. Let $\xi_1, \ldots, \xi_n$ be a collection of 1-forms holomorphic at the points P, P', Q. Then the following statement holds:*

$$\int_P^Q \xi_1 \cdots \xi_n = \sum_{i=0}^{n} \int_{P'}^{Q} \xi_1 \cdots \xi_i \int_P^{P'} \xi_{i+1} \cdots \xi_n.$$

Proof. We proceed by induction. The case $n = 1$ is clear. Let us suppose the statement holds for $n = k$. Then we have that

$$\begin{aligned} \int_P^Q \xi_1 \cdots \xi_{k+1} &= \left(\int_P^Q \xi_1 \cdots \xi_k\right)(R) \int_P^R \xi_{k+1} \\ &= \left(\sum_{i=0}^{k} \int_{P'}^{Q} \xi_1 \cdots \xi_i \int_P^{P'} \xi_{i+1} \cdots \xi_k\right)(R) \int_P^R \xi_{k+1} \\ &= \left(\int_P^{P'} \xi_1 \cdots \xi_k\right)(R) \int_P^R \xi_{k+1} \qquad (4.2) \\ &\quad + \left(\int_{P'}^{Q} \xi_1\right)\left(\int_P^{P'} \xi_2 \cdots \xi_k\right)(R) \int_P^R \xi_{k+1} \qquad (4.3) \\ &\quad \cdots + \left(\int_{P'}^{Q} \xi_1 \cdots \xi_{k-1} \int_P^{P'} \xi_k\right)(R) \int_P^R \xi_{k+1} \qquad (4.4) \\ &\quad + \left(\int_{P'}^{Q} \xi_1 \cdots \xi_k\right)(R) \int_P^R \xi_{k+1}. \qquad (4.5) \end{aligned}$$

Observe that this last iterated integral (4.5) can be rewritten as

$$\left(\int_{P'}^{Q} \xi_1 \cdots \xi_k\right)(R) \left(\int_P^{P'} \xi_{k+1} + \int_{P'}^{R} \xi_{k+1}\right),$$

and that further, the terms from (4.2) through (4.4) give us

$$\sum_{i=0}^{k-1} \int_{P'}^{Q} \xi_1 \cdots \xi_i \int_{P}^{P'} \xi_{i+1} \cdots \xi_{k+1},$$

Thus we have

$$\begin{aligned}\int_P^Q \xi_1 \cdots \xi_{k+1} &= \sum_{i=0}^{k-1} \int_{P'}^{Q} \xi_1 \cdots \xi_i \int_{P}^{P'} \xi_{i+1} \cdots \xi_{k+1} \\ &\quad + \left(\int_{P'}^{Q} \xi_1 \cdots \xi_k \right) \left(\int_{P}^{P'} \xi_{k+1} \right) + \int_{P'}^{Q} \xi_1 \cdots \xi_{k+1} \\ &= \sum_{i=0}^{k+1} \int_{P'}^{Q} \xi_1 \cdots \xi_i \int_{P}^{P'} \xi_{i+1} \cdots \xi_{k+1},\end{aligned}$$

as desired. □

Applying Lemma 4.3 twice, we obtain the following, which will be used to link integrals between different residue disks:

Lemma 4.4 (Link lemma)**.** *Let points P, P', Q', Q be on C such that a path is to be taken from P to P' to Q' to Q. Let $\xi_1, \ldots, \xi_n$ be a collection of 1-forms holomorphic at the points P, P', Q, Q'. Then we have*

$$\int_P^Q \xi_1 \cdots \xi_n = \sum_{i=0}^{n} \int_{Q'}^{Q} \xi_1 \cdots \xi_i \left(\sum_{j=i}^{n} \int_{P'}^{Q'} \xi_{i+1} \cdots \xi_j \int_{P}^{P'} \xi_{j+1} \cdots \xi_n \right).$$

Below we record aspecific case of the link lemma, which we shall use throughout this paper.

Example 4.5 (Link lemma for double integrals)**.** Suppose we have two differentials ξ_0, ξ_1. Then we have

$$\int_P^Q \xi_0 \xi_1 = \int_P^{P'} \xi_0 \xi_1 + \int_{P'}^{Q'} \xi_0 \xi_1 + \int_{Q'}^{Q} \xi_0 \xi_1 + \int_P^{P'} \xi_1 \int_{P'}^{Q} \xi_0 + \int_{P'}^{Q'} \xi_1 \int_{Q'}^{Q} \xi_0.$$

5. Tiny iterated integrals

We begin with an algorithm to compute tiny iterated integrals.

Algorithm 5.1 (Tiny iterated integrals)**.**
Input: Points $P, Q \in C(\mathbb{C}_p)$ in the same residue disk (neither equal to the point at infinity) and differentials $\xi_1, \ldots, \xi_n$ without poles in the disk of P.
Output: The integral $\int_P^Q \xi_1 \xi_2 \cdots \xi_n$.

(1) Compute a parametrization $(x(t), y(t))$ at P in terms of a local coordinate t.
(2) For each k, write $\xi_k(x, y)$ in terms of t: $\xi_k(t) := \xi_k(x(t), y(t))$.
(3) Let $I_{n+1}(t) := 1$.

(4) Compute, for $k = n, \dots, 2$, in descending order,

$$\begin{aligned} I_k(t) &= \int_P^{R_{k-1}} \xi_k I_{k+1} \\ &= \int_0^{t(R_{k-1})} \xi_k(u) I_{k+1}(u), \end{aligned}$$

with R_{k-1} in the disc of P.

(5) Upon computing $I_2(t)$, we arrive at the desired integral:

$$\int_P^Q \xi_1 \xi_2 \cdots \xi_n = I_1(t) = \int_0^{t(Q)} \xi_1(u) I_2(u).$$

We show how we carry out Algorithm 5.1 for double integrals on an elliptic curve.

Example 5.2 (A tiny double integral)**.** Let C be the elliptic curve $y^2 = x(x-1)(x+9)$, let $p = 7$, and consider the points $P = (9, 36), Q = \phi(P)$, and $R = (a + x(P), \sqrt{f(a + x(P))})$ so that R is in the same disk as P and Q. Furthermore, let $\omega_0 = \frac{dx}{2y}, \omega_1 = \frac{x dx}{2y}$. We compute the double integral $\int_P^Q \omega_0 \omega_1$.

First compute the local coordinates at P:

$$\begin{aligned} x(t) &= 9 + t + O(t^{20}) \\ y(t) &= 36 + \frac{21}{4}t + \frac{119}{1152}t^2 - \frac{65}{55296}t^3 + \frac{2219}{95551488}t^4 - \frac{7}{509607936}t^5 + O(t^6). \end{aligned}$$

Then setting $I_2 := \int x \frac{dx}{2y}$, and making it a definite integral, we have

$$\begin{aligned} I_2|_P^R &= \int_P^R x \frac{dx}{2y} \\ &= \int_0^a x(t) \frac{dx(t)}{2y(t)} \\ &= \frac{1}{8}a - \frac{5}{2304}a^2 + \frac{91}{995328}a^3 - \frac{1121}{191102976}a^4 + \frac{22129}{45864714240}a^5 \\ &\qquad - \frac{360185}{7925422620672}a^6 + \frac{36737231}{7988826001637376}a^7 + O(a^8), \end{aligned}$$

from which we arrive at

$$\begin{aligned} I &= \int_0^{x(Q)-x(P)} I_2(a) \frac{dx(R(a))}{2y(R(a))} \\ &= 4 \cdot 7^2 + 5 \cdot 7^3 + 2 \cdot 7^5 + 4 \cdot 7^6 + 2 \cdot 7^7 + O(7^8). \end{aligned}$$

6. Iterated integrals: linear system

As in the case of computing single integrals, to compute general iterated Coleman integrals, we use Kedlaya's algorithm to calculate the action of Frobenius on de Rham cohomology. This gives us a linear system that allows us to solve for all $(2g)^n$ n-fold iterated integrals on basis differentials.

Theorem 6.1. *Let $P, Q \in C(\mathbb{C}_p)$ be non-Weierstrass points such that the residue fields of P, Q are contained in $\mathbb{F}_{p^m}$. Let M be the matrix of the action of the m-th power of Frobenius on the basis differentials $\omega_0, \dots, \omega_{2g-1}$. For constants $c_{i_0,\dots,i_{n-1}}$*

computable in terms of $(n-1)$-fold iterated integrals and n-fold tiny iterated integrals, the n-fold iterated Coleman integrals on basis differentials between P, Q can be computed via a linear system of the form

$$\begin{pmatrix} \vdots \\ \int_P^Q \omega_{i_0} \cdots \omega_{i_{n-1}} \\ \vdots \end{pmatrix} = (I_{(2g)^n \times (2g)^n} - (M^t)^{\otimes n})^{-1} \begin{pmatrix} \vdots \\ c_{i_0 \cdots i_{n-1}} \\ \vdots \end{pmatrix}.$$

Proof. By the Link lemma (Lemma 4.4), we can reduce to the case where both P and Q are Teichmüller points (points fixed by some power of ϕ). Then we have

$$\begin{aligned} \int_P^Q \omega_{i_i} \cdots \omega_{i_n} &= \int_{\phi^m(P)}^{\phi^m(Q)} \omega_{i_i} \cdots \omega_{i_n} \\ &= \int_P^Q (\phi^m)^*(\omega_{i_i} \cdots \omega_{i_n}) \\ &= \int_P^Q (\phi^m)^*(\omega_{i_i}) \cdots (\phi^m)^*(\omega_{i_n}). \end{aligned} \tag{6.1}$$

Recall that given $\omega_0, \ldots, \omega_{2g-1}$ a basis for $H^1_{dR}(C')$, we have

$$(\phi^m)^* \omega_{i_\ell} = df_{i_\ell} + \sum_{j=0}^{2g-1} M_{i_\ell j} \omega_j.$$

Substituting this expression in for each factor of (6.1) and expanding yields the linear system. □

To illustrate our methods, in the next section, we present a more explicit version of this theorem, accompanied by algorithms, in the case of double integrals. We show how these are used in Kim's nonabelian Chabauty method in Section 8.

7. Explicit double integrals

7.1. **The linear system for double integrals between Teichmüller points.** In this subsection, we make explicit one aspect of Theorem 6.1: we give an algorithm to compute double integrals between Teichmüller points.

Algorithm 7.1 (Double Coleman integration between Teichmüller points)**.**
Input: The basis differentials $(\omega_i)_{i=0}^{2g-1}$, Teichmüller points $P, Q \in C(\mathbb{C}_p)$ in non-Weierstrass residue disks, and a positive integer m such that the residue fields of P, Q are contained in $\mathbb{F}_{p^m}$.

Output: The double integrals $\left(\int_P^Q \omega_i \omega_j\right)_{i,j=0}^{2g-1}$.

(1) Calculate the action of the m-th power of Frobenius on each basis element: $(\phi^m)^* \omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij} \omega_j$.
(2) Use Algorithm 4.1 to compute the single Coleman integrals $\int_P^Q \omega_j$ on all basis differentials.
(3) Use Step 2 and linearity to recover the other single Coleman integrals: $\int_P^Q df_i f_k$, $\int_P^Q \sum_{j=0}^{2g-1} M_{ij} \omega_j f_k$ for each i, k.

(4) Use the results of the above two steps to write down, for each i, k, the constant

$$c_{ik} = \int_P^Q df_i(R)(f_k(R)) - f_k(P)(f_i(Q) - f_i(P)) + \int_P^Q \sum_{j=0}^{2g-1} M_{ij}\omega_j(R)(f_k(R) - f_k(P)) + f_i(Q)\int_P^Q \sum_{j=0}^{2g-1} M_{kj}\omega_j - \int_P^Q f_i(R)(\sum_{j=0}^{2g-1} M_{kj}\omega_j(R)).$$

(5) Recover the double integrals (see Remark 7.2 below) via the linear system

$$\begin{pmatrix} \int_P^Q \omega_0\omega_0 \\ \int_P^Q \omega_0\omega_1 \\ \vdots \\ \int_P^Q \omega_{2g-1}\omega_{2g-1} \end{pmatrix} = (I_{4g^2\times 4g^2} - (M^t)^{\otimes 2})^{-1} \begin{pmatrix} c_{00} \\ c_{01} \\ \vdots \\ c_{2g-1,2g-1} \end{pmatrix}.$$

Remark 7.2. We obtain the linear system in the following manner. Since P, Q are Teichmüller, we have

$$\int_P^Q \omega_i\omega_k = \int_{\phi^m(P)}^{\phi^m(Q)} \omega_i\omega_k = \int_P^Q (\phi^m)^*(\omega_i\omega_k). \tag{7.1}$$

We begin by expanding the right side of (7.1).

Recall that given $\omega_0, \ldots, \omega_{2g-1}$ a basis for $H^1_{dR}(C')$, we have

$$(\phi^m)^*\omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j.$$

Thus we have

$$\begin{aligned}\int_P^Q (\phi^m)^*(\omega_i\omega_k) &= \int_P^Q (\phi^m)^*(\omega_i)(\phi^m)^*(\omega_k) \\ &= \int_P^Q (df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j)(df_k + \sum_{j=0}^{2g-1} M_{kj}\omega_j) \\ &= \int_P^Q df_i df_k + (\sum_{j=0}^{2g-1} M_{ij}\omega_j)df_k + df_i(\sum_{j=0}^{2g-1} M_{kj}\omega_j) + (\sum_{j=0}^{2g-1} M_{ij}\omega_j)(\sum_{j=0}^{2g-1} M_{kj}\omega_j)\end{aligned}$$

We expand the first three quantities separately. First, we have

$$\begin{aligned}\int_P^Q df_i df_k &= \int_P^Q df_i(R)\int_P^R df_k \\ &= \int_P^Q df_i(R)(f_k(R) - f_k(P)) \\ &= \int_P^Q df_i(R)(f_k(R)) - f_k(P)\int_P^Q df_i(R) \\ &= \int_P^Q df_i(R)(f_k(R)) - f_k(P)(f_i(Q) - f_i(P)).\end{aligned}$$

Next, we have

$$\begin{aligned}\int_P^Q (\sum_{j=0}^{2g-1} M_{ij}\omega_j) df_k &= \int_P^Q \sum_{j=0}^{2g-1} M_{ij}\omega_j(R) \int_P^R df_k \\ &= \int_P^Q \sum_{j=0}^{2g-1} M_{ij}\omega_j(R)(f_k(R) - f_k(P)).\end{aligned}$$

The third term (via integration by parts) is

$$\begin{aligned}\int_P^Q df_i(\sum_{j=0}^{2g-1} M_{kj}\omega_j) &= \int_P^Q df_i(R) \int_P^R (\sum_{j=0}^{2g-1} M_{kj}\omega_j) \\ &= \left(f_i(R) \int_P^R (\sum_{j=0}^{2g-1} M_{kj}\omega_j) \right) |_{R=P}^{R=Q} - \int_P^Q f_i(R)(\sum_{j=0}^{2g-1} M_{kj}\omega_j(R)) \\ &= f_i(Q) \int_P^Q \sum_{j=0}^{2g-1} M_{kj}\omega_j - \int_P^Q f_i(R)(\sum_{j=0}^{2g-1} M_{kj}\omega_j(R)).\end{aligned}$$

Denote the sum of these terms by c_{ik}; in other words,

$$\begin{aligned}c_{ik} = &\int_P^Q df_i(R)(f_k(R)) - f_k(P)(f_i(Q) - f_i(P)) + \int_P^Q \sum_{j=0}^{2g-1} M_{ij}\omega_j(R)(f_k(R) - f_k(P)) \\ &+ f_i(Q) \int_P^Q \sum_{j=0}^{2g-1} M_{kj}\omega_j - \int_P^Q f_i(R)(\sum_{j=0}^{2g-1} M_{kj}\omega_j(R)).\end{aligned}$$

Then rearranging terms, our linear system reads

$$\begin{pmatrix} \int_P^Q \omega_0\omega_0 \\ \int_P^Q \omega_0\omega_1 \\ \vdots \\ \int_P^Q \omega_{2g-1}\omega_{2g-1} \end{pmatrix} = (I_{4g^2\times 4g^2} - (M^t)^{\otimes 2})^{-1} \begin{pmatrix} c_{00} \\ c_{01} \\ \vdots \\ c_{2g-1,2g-1} \end{pmatrix}.$$

7.2. **Linking double integrals.** Let P' and Q' be in the disks of P and Q, respectively. Using the Link lemma for double integrals (Example 4.5), we may link double integrals between different residue disks:

(7.2)

$$\int_P^Q \omega_i\omega_k = \int_P^{P'} \omega_i\omega_k + \int_{P'}^{Q'} \omega_i\omega_k + \int_{Q'}^{Q} \omega_i\omega_k + \int_P^{P'} \omega_k \int_{P'}^{Q} \omega_i + \int_{P'}^{Q'} \omega_k \int_{Q'}^{Q} \omega_i.$$

Algorithm 7.3 (Double Coleman integration using intermediary Teichmüller points)**.**

Input: The basis differentials $(\omega_i)_{i=0}^{2g-1}$, points $P, Q \in C(\mathbb{C}_p)$ in non-Weierstrass residue disks.

Output: The double integrals $\left(\int_P^Q \omega_i\omega_j\right)_{i,j=0}^{2g-1}$.

(1) Compute Teichmüller points P', Q' in the disks of P, Q, respectively.

(2) Use Algorithm 4.1 to compute the single integrals $\int_P^Q \omega_i, \int_{P'}^P \omega_i, \int_Q^{Q'} \omega_i$ for all i.
(3) Use Algorithm 5.1 to compute the tiny double integrals $\int_{P'}^P \omega_i\omega_k, \int_{Q'}^Q \omega_i\omega_k$.
(4) Use Algorithm 7.1 to compute the double integrals $\{\int_{P'}^{Q'} \omega_i\omega_j\}_{i,j=0}^{2g-1}$.
(5) Correct endpoints using

$$\int_P^Q \omega_i\omega_k = \int_P^{P'} \omega_i\omega_k + \int_{P'}^{Q'} \omega_i\omega_k + \int_{Q'}^Q \omega_i\omega_k + \int_P^{P'} \omega_k \int_{P'}^Q \omega_i + \int_{P'}^{Q'} \omega_k \int_{Q'}^Q \omega_i.$$

7.3. **Without Teichmüller points.** Alternatively, instead of finding Teichmüller points and correcting endpoints, we can directly compute double integrals using a slightly different linear system. Indeed, using the Link lemma for double integrals, we take $\phi(P)$ and $\phi(Q)$ to be the points in the disks of P and Q, respectively, which gives

(7.3)
$$\int_P^Q \omega_i\omega_k = \int_P^{\phi(P)} \omega_i\omega_k + \int_{\phi(P)}^{\phi(Q)} \omega_i\omega_k + \int_{\phi(Q)}^Q \omega_i\omega_k + \int_P^{\phi(P)} \omega_k \int_{\phi(P)}^Q \omega_i + \int_{\phi(P)}^{\phi(Q)} \omega_k \int_{\phi(Q)}^Q \omega_i$$

To write down a linear system without Teichmüller points, we begin as before, with

$$\int_{\phi(P)}^{\phi(Q)} \omega_i\omega_k = \int_P^Q \phi^*(\omega_i\omega_k) = c_{ik} + \int_P^Q \left(\sum_{j=0}^{2g-1} A_{ij}\omega_j\right)\left(\sum_{j=0}^{2g-1} A_{kj}\omega_j\right). \tag{7.4}$$

Putting together (7.3) and (7.4), we get
(7.5)

$$\begin{pmatrix} \vdots \\ \int_P^Q \omega_i\omega_k \\ \vdots \end{pmatrix} = (I_{4g^2\times 4g^2} - (M^t)^{\otimes 2})^{-1} \begin{pmatrix} \vdots \\ c_{ik} - \int_{\phi(P)}^P \omega_i\omega_k - \left(\int_P^Q \omega_i\right)\left(\int_{\phi(P)}^P \omega_k\right) \\ - \left(\int_Q^{\phi(Q)} \omega_i\right)\left(\int_{\phi(P)}^{\phi(Q)} \omega_k\right) + \int_{\phi(Q)}^Q \omega_i\omega_k \\ \vdots \end{pmatrix}.$$

This gives us the following alternative to Algorithm 7.1:

Algorithm 7.4 (Double Coleman integration)**.**
Input: The basis differentials $(\omega_i)_{i=0}^{2g-1}$, points $P, Q \in C(\mathbb{Q}_p)$ in non-Weierstrass residue disks or in Weierstrass disks in the region of convergence.
Output: The double integrals $\left(\int_P^Q \omega_i\omega_j\right)_{i,j=0}^{2g-1}$.

(1) Use Algorithm 4.1 to compute the single integrals $\int_P^Q \omega_i, \int_{\phi(P)}^{\phi(Q)} \omega_i$ for all i.
(2) Use Algorithm 5.1 to compute $\int_{\phi(P)}^P \omega_i\omega_k, \int_{\phi(Q)}^Q \omega_i\omega_k$ for all i, k
(3) As in Step 4 of Algorithm 7.1, compute the constants c_{ik} for all i, k.

(4) Recover the double integrals using the linear system

$$\begin{pmatrix} \vdots \\ \int_P^Q \omega_i\omega_k \\ \vdots \end{pmatrix} = (I_{4g^2\times 4g^2} - (M^t)^{\otimes 2})^{-1} \begin{pmatrix} \vdots \\ c_{ik} - \int_{\phi(P)}^P \omega_i\omega_k - \left(\int_P^Q \omega_i\right)\left(\int_{\phi(P)}^P \omega_k\right) \\ - \left(\int_Q^{\phi(Q)} \omega_i\right)\left(\int_{\phi(P)}^{\phi(Q)} \omega_k\right) + \int_{\phi(Q)}^Q \omega_i\omega_k \\ \vdots \end{pmatrix}$$

Example 7.5. Let C be the genus 2 curve $y^2 = x^5 - x^4 + x^3 + x^2 - 2x + 1$ and let $P = (1, -1), Q = (-1, -1)$ and $p = 7$. We compute double integrals on basis differentials:

$$\int_P^Q \omega_0\omega_0 = 2\cdot 7^2 + 7^3 + 4\cdot 7^4 + O(7^5)$$
$$\int_P^Q \omega_0\omega_1 = 7^2 + 5\cdot 7^3 + 3\cdot 7^4 + O(7^5)$$
$$\int_P^Q \omega_0\omega_2 = 4\cdot 7 + 5\cdot 7^2 + 7^3 + O(7^4)$$
$$\int_P^Q \omega_0\omega_3 = 7 + 5\cdot 7^2 + 3\cdot 7^4 + O(7^5)$$
$$\int_P^Q \omega_1\omega_0 = 7^2 + 6\cdot 7^3 + 5\cdot 7^4 + O(7^5)$$
$$\int_P^Q \omega_1\omega_1 = 4\cdot 7^2 + 3\cdot 7^3 + O(7^5)$$
$$\int_P^Q \omega_1\omega_2 = 5\cdot 7 + 6\cdot 7^2 + 2\cdot 7^3 + 4\cdot 7^4 + O(7^5)$$
$$\int_P^Q \omega_1\omega_3 = 2 + 3\cdot 7 + 7^2 + 4\cdot 7^3 + O(7^4)$$
$$\int_P^Q \omega_2\omega_0 = 7^2 + 4\cdot 7^3 + O(7^4)$$
$$\int_P^Q \omega_2\omega_1 = 4\cdot 7 + 6\cdot 7^2 + 4\cdot 7^3 + 5\cdot 7^4 + O(7^5)$$
$$\int_P^Q \omega_2\omega_2 = 2 + 5\cdot 7 + 3\cdot 7^2 + O(7^3)$$
$$\int_P^Q \omega_2\omega_3 = 5 + 2\cdot 7 + 3\cdot 7^2 + O(7^3)$$
$$\int_P^Q \omega_3\omega_0 = 3\cdot 7 + 2\cdot 7^2 + 5\cdot 7^3 + 5\cdot 7^4 + O(7^5)$$
$$\int_P^Q \omega_3\omega_1 = 5 + 5\cdot 7 + 7^2 + 6\cdot 7^3 + O(7^4)$$
$$\int_P^Q \omega_3\omega_2 = 6 + 7 + 5\cdot 7^2 + O(7^3)$$
$$\int_P^Q \omega_3\omega_3 = 2 + 6\cdot 7 + 5\cdot 7^2 + O(7^3)$$

Example 7.6. Using the previous example, we verify the Fubini identity

$$\int_P^Q \omega_j\omega_i + \int_P^Q \omega_i\omega_j = \left(\int_P^Q \omega_i\right)\left(\int_P^Q \omega_j\right).$$

We have

$$\begin{aligned}
\int_P^Q \omega_0 &= 5\cdot 7 + 2\cdot 7^2 + 5\cdot 7^3 + 7^4 + 4\cdot 7^5 + O(7^6)\\
\int_P^Q \omega_1 &= 6\cdot 7 + 6\cdot 7^2 + 2\cdot 7^3 + 4\cdot 7^4 + 3\cdot 7^5 + O(7^6)\\
\int_P^Q \omega_2 &= 5 + 5\cdot 7^3 + 6\cdot 7^4 + 2\cdot 7^5 + O(7^6)\\
\int_P^Q \omega_3 &= 5 + 3\cdot 7 + 4\cdot 7^2 + 3\cdot 7^3 + 6\cdot 7^4 + 2\cdot 7^5 + O(7^6).
\end{aligned}$$

We see, for example,

$$\int_P^Q \omega_0\omega_1 + \int_P^Q \omega_1\omega_0 = 2\cdot 7^2 + 4\cdot 7^3 + 2\cdot 7^4 + O(7^5) = \left(\int_P^Q \omega_0\right)\left(\int_P^Q \omega_1\right)$$

$$\int_P^Q \omega_2\omega_3 + \int_P^Q \omega_3\omega_2 = 4 + 4\cdot 7 + 7^2 + O(7^3) = \left(\int_P^Q \omega_2\right)\left(\int_P^Q \omega_3\right).$$

7.4. **Weierstrass points.** Suppose one of P or Q is a finite Weierstrass point. Then directly using the linear system as above fails, since the f_i have essential singularities at finite Weierstrass points. We remedy this as follows:

Proposition 7.7. *Let Q be a non-Weierstrass point, P a finite Weierstrass point, and S be a point in the residue disk of P, near the boundary. Then the integral from P to Q can be computed as a sum of integrals:*

$$\int_P^Q \omega_i\omega_k = \int_P^S \omega_i\omega_k + \int_S^Q \omega_i\omega_k + \int_P^S \omega_k \int_S^Q \omega_i.$$

Proof. This follows from Lemma 4.3 in the case of $n = 2$, where $P' = S$. □

To compute tiny iterated integrals in a Weierstrass disk, we slightly modify Algorithm 5.1:

Algorithm 7.8 (Tiny iterated integral in a Weierstrass disk)**.**
Input: P a Weierstrass point, d the degree of totally ramified extension, ω_i, ω_j basis differentials
Output: The integral

$$\int_P^S \omega_i\omega_j = \int_P^S \omega_i(R) \int_P^R \omega_j = \int_{t=0}^{t=1} \omega_i(R) \int_{u=0}^{u=t} \omega_j.$$

(1) Compute local coordinates $(x(u), u)$ at P.
(2) Let $a = p^{1/d}$. Rescale coordinates so that $y := au, x := x(au)$.
(3) Compute $I_2(u) = \int x^j \frac{dx}{2y}$ as a power series in u.
(4) Compute the appropriate definite integral using the step above:

$$\int_R^S x^j \frac{dx}{2y} = \int_0^t x(au)\frac{adu}{u} = I_2(t)$$

(where $R = (x(t), t)$). Call this definite integral (now a power series in t) I_2.

(5) Now since $R = (x(t), t)$, we have $\int_P^S \omega_i\omega_j = \int_0^1 x(t)^i I_2 \frac{dx(t)}{2t}$.

Suppose P is a finite Weierstrass point. While one could compute the integral $\int_P^Q \omega_i\omega_j$ directly using Algorithm 7.4 for all of the tiny double integrals (and Algorithm 7.8 for the other double integrals), in practice, that approach is expensive, as it requires the computation of several intermediate integrals with Frobenius of points that are defined over ramified extensions. This, in turn, makes the requisite degree d extension for convergence quite large.

Instead, the key idea is to compute a local parametrization at the finite Weierstrass point P and to use this to compute the indefinite integral $\int_P^* \omega_i$. Then to compute integrals involving "boundary points," one can simply evaluate this indefinite integral at the appropriate points, instead of directly computing parametrizations, and thus integrals, over a totally ramified extension of $\mathbb{Q}_p$. This idea is also used to evaluate double integrals involving boundary points.

Algorithm 7.9 (Intermediary integrals for double integrals with a Weierstrass endpoint)**.**
Input: P finite Weierstrass point, Q non-Weierstrass point, d the degree of totally ramified extension, n the precision of $\mathbb{Q}_p$, basis differentials ω_i, ω_j.
Output: Necessary things for the eventual computation of $\int_P^Q \omega_i\omega_j$.

(1) Compute $(x(t), t)$ local coordinates at P to precision nd.
(2) Let $S = (x(a), a)$, where $a = p^{1/d}$.
(3) Compute as a power series in t, $I_2(t) = \int x(t)^i \frac{dx(t)}{y(t)}$.
(4) Compute the definite integral $\int_P^S \omega_i = I_2(a)$.
(5) For all $i < j$, compute the definite integral $\int_P^S \omega_i\omega_j$ via Algorithm 5.1. Keep the intermediary indefinite integral.
(6) For all $i = j$, use the fact that $\int_P^S \omega_i\omega_j = \frac{1}{2}\left(\int_P^S \omega_i\right)^2$ to compute the double integral in terms of the single integral.
(7) For all $i > j$, use the fact that $\int_P^S \omega_i\omega_j = -\int_P^S \omega_j\omega_i + \int_P^S \omega_i \int_P^S \omega_j$ to compute $\int_P^S \omega_i\omega_j$ (instead of directly computing it as a double integral).
(8) Compute $\int_S^{\phi(S)} \omega_i = \int_P^{\phi(S)} \omega_i - \int_P^S \omega_i$ by the indefinite integral in Step 3. Use this to deduce $\int_S^{\phi(S)} \omega_i\omega_j$ for $i = j$.
(9) Use the indefinite integral in Step 5 to get $\int_S^{\phi(S)} \omega_i\omega_j$ for $i < j$.
(10) Repeat the trick in Step 7 to get $\int_S^{\phi(S)} \omega_i\omega_j$ for $i > j$.
(11) Compute $\int_Q^{\phi(Q)} \omega_i$ and use it to deduce $\int_Q^{\phi(Q)} \omega_i\omega_j$ for $i = j$.
(12) Compute $\int_Q^{\phi(Q)} \omega_i\omega_j$ for $i < j$.
(13) Repeat the trick in Step 7 to get $\int_Q^{\phi(Q)} \omega_i\omega_j$ for $i < j$.
(14) Use $\int_S^Q \omega_i = \int_P^Q \omega_i - \int_P^S \omega_i$ to get $\int_S^Q \omega_i$.

Algorithm 7.10 (Double integrals from a Weierstrass endpoint)**.**
Input: P finite Weierstrass point, Q non-Weierstrass point, ω_i, ω_j basis differentials.
Output: The double integrals $\int_P^Q \omega_i\omega_j$.

(1) Compute all of the integrals as in Algorithm 7.9.
(2) Compute double integrals $\int_S^Q \omega_i\omega_j$ using the terms in Step 1 as appropriate in Algorithm 7.4. (See Remark 7.11 for an additional improvement to this step)
(3) Use additivity to recover the double integrals $\int_P^Q \omega_i\omega_j = \int_P^S \omega_i\omega_j + \int_S^Q \omega_i\omega_j + \int_P^S \omega_i \int_S^Q \omega_j$.

Remark 7.11. Note that in the case of $g = 1$, the linear system only yields *one* double integral not obtainable through single integrals. Indeed, for $0 \leq i, j \leq 1$, we have $\int_S^Q \omega_i\omega_i = \frac{1}{2}\left(\int_S^Q \omega_i\right)^2$ and $\int_S^Q \omega_i\omega_j = -\int_S^Q \omega_j\omega_i + \int_S^Q \omega_i \int_S^Q \omega_j$. So it suffices to compute $\int_S^Q \omega_0\omega_1$. Thus, rather than computing all of the constants $c_{00}, c_{01}, c_{10}, c_{11}$ and their correction factors (see (7.5)), if we pre-compute the two double integrals that are expressible in terms of single integrals, as well as the product of single integrals that relates $\int_S^Q \omega_1\omega_0$ to $\int_S^Q \omega_0\omega_1$, it suffices to compute c_{01} (and its correction factor) to solve for the other three constants and $\int_S^Q \omega_0\omega_1$.

In other words, the linear system in Algorithm 7.4 tells us that

$$(I_{4\times 4} - (M^t)^{\otimes 2}) \begin{pmatrix} \vdots \\ \int_P^Q \omega_i\omega_k \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ c_{ik} - \int_{\phi(P)}^P \omega_i\omega_k - \left(\int_P^Q \omega_i\right)\left(\int_{\phi(P)}^P \omega_k\right) \\ -\left(\int_Q^{\phi(Q)} \omega_i\right)\left(\int_{\phi(P)}^{\phi(Q)} \omega_k\right) + \int_{\phi(Q)}^Q \omega_i\omega_k \\ \vdots \end{pmatrix},$$

which we write as

$$A \begin{pmatrix} i_{00} \\ v_{01} \\ s_{01} - v_{01} \\ i_{11} \end{pmatrix} = \begin{pmatrix} x_{00} \\ \ell_{01} \\ x_{10} \\ x_{11} \end{pmatrix},$$

where the vector on the left consists of integrals (with $i_{00} = \int_S^Q \omega_0\omega_0, i_{11} = \int_S^Q \omega_1\omega_1, s_{01} = \int_S^Q \omega_0 \int_S^Q \omega_1$ all computed), and the vector on the right consists of constants (with ℓ_{01} computed). So we solve for $x_{00}, x_{10}, x_{11}, v_{01}$, since knowing $v_{01} = \int_S^Q \omega_0\omega_1$ gives us the complete set of double integrals on basis differentials:

$$\begin{pmatrix} x_{00} \\ x_{10} \\ x_{11} \\ v_{01} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -(a_{01} - a_{01}) \\ 0 & 0 & 0 & -(a_{11} - a_{12}) \\ 0 & 1 & 0 & -(a_{21} - a_{22}) \\ 0 & 0 & 1 & -(a_{31} - a_{32}) \end{pmatrix}^{-1} \left(A \begin{pmatrix} i_{00} \\ 0 \\ c_{01} \\ i_{11} \end{pmatrix} - \begin{pmatrix} 0 \\ \ell_{01} \\ 0 \\ 0 \end{pmatrix}, \right)$$

where $A = (a_{ij})$. While this only gives a constant speed-up in terms of complexity, in practice, this helps when S is defined over a highly ramified extension of $\mathbb{Q}_p$.

As numerical checks, one may use the following corollaries of Proposition 7.7.

Corollary 7.12. *For P, Q Weierstrass points and S a third point, we have additivity in endpoints: $\int_P^Q \omega_i\omega_j + \int_Q^S \omega_i\omega_j = \int_P^S \omega_i\omega_j$.*

Corollary 7.13. *For P, Q Weierstrass points, we have $\int_P^Q \omega_i\omega_j + \int_P^Q \omega_j\omega_i = 0$.*

It is worth noting that in general, unlike in the case of a single Coleman integral, for P and Q both Weierstrass points, unless $i = k$, the double Coleman integral $\int_P^Q \omega_i\omega_k$ is not necessarily 0. However, in the case of $i = k$, the integral can be computed as $\int_P^Q \omega_i\omega_i = \frac{1}{2}\left(\int_P^Q \omega_i\right)^2 = 0$.

Example 7.14. Consider the curve $y^2 = x(x-1)(x+9)$, over $\mathbb{Q}_7$, and the points $P_1 = (1,0)$, $P_2 = (0,0)$, and $Q = (-1,4)$. We have

$$\begin{pmatrix} \int_{P_1}^Q \omega_0\omega_0 \\ \int_{P_1}^Q \omega_0\omega_1 \\ \int_{P_1}^Q \omega_1\omega_0 \\ \int_{P_1}^Q \omega_1\omega_1 \end{pmatrix} = \begin{pmatrix} 2\cdot 7^2 + 5\cdot 7^3 + 4\cdot 7^4 + 3\cdot 7^5 + O(7^6) \\ 6\cdot 7 + 5\cdot 7^2 + 4\cdot 7^3 + 6\cdot 7^4 + O(7^6) \\ 2\cdot 7^2 + 3\cdot 7^3 + 3\cdot 7^4 + 7^5 + O(7^6) \\ 1 + 5\cdot 7 + 5\cdot 7^3 + 4\cdot 7^4 + 4\cdot 7^5 + O(7^6) \end{pmatrix}$$

and

$$\begin{pmatrix} \int_{P_2}^Q \omega_0\omega_0 \\ \int_{P_2}^Q \omega_0\omega_1 \\ \int_{P_2}^Q \omega_1\omega_0 \\ \int_{P_2}^Q \omega_1\omega_1 \end{pmatrix} = \begin{pmatrix} 2\cdot 7^2 + 5\cdot 7^3 + 4\cdot 7^4 + 3\cdot 7^5 + O(7^6) \\ 2\cdot 7^2 + 7^3 + 6\cdot 7^4 + 5\cdot 7^5 + O(7^6) \\ 6\cdot 7 + 5\cdot 7^2 + 6\cdot 7^3 + 3\cdot 7^4 + 3\cdot 7^5 + O(7^6) \\ 1 + 5\cdot 7 + 5\cdot 7^3 + 4\cdot 7^4 + 4\cdot 7^5 + O(7^6) \end{pmatrix},$$

from which we see that $\int_{P_1}^{P_2} \omega_0\omega_1 \neq 0$ and likewise $\int_{P_1}^{P_2} \omega_1\omega_0 \neq 0$.

8. Kim's nonabelian Chabauty method

We now present the motivation for all of the algorithms thus far. Let $\mathcal{C}/\mathbb{Z}$ be the minimal regular model of an elliptic curve $C/\mathbb{Q}$ of analytic rank 1 with Tamagawa numbers all 1. Let $\mathcal{X} = \mathcal{C} - \{\infty\}$ and $\omega_0 = \frac{dx}{2y}, \omega_1 = \frac{xdx}{2y}$. Taking a tangential basepoint b at ∞ (or letting b be an integral 2-torsion point), we have the analytic functions

$$\log_{\omega_0}(z) = \int_b^z \omega_0, \quad D_2(z) = \int_b^z \omega_0\omega_1.$$

With this setup, we have

Theorem 8.1 ([3, 10])**.** *Suppose P is a point of infinite order in $\mathcal{C}(\mathbb{Z})$. Then $\mathcal{X}(\mathbb{Z}) \subset \mathcal{C}(\mathbb{Z}_p)$ is in the zero set of*

$$f(z) := (\log_{\omega_0}(P))^2 D_2(z) - (\log_{\omega_0}(z))^2 D_2(P).$$

Corollary 8.2 ([3, 10])**.** *The expression*

$$\frac{D_2(P)}{(\log_{\omega_0}(P))^2} \tag{8.1}$$

is independent of the point P of infinite order in $\mathcal{C}(\mathbb{Z})$.

Example 8.3. We revisit Example 1 in [3]. Let E be the rank 1 elliptic curve $y^2 = x^3 - 1323x + 3942$, with minimal model $\mathcal{E}$ having Cremona label '65a1'. Consider the following points on E which are integral on $\mathcal{E}$: $b = (3,0), P = (39,108), Q = (-33,-108), R = (147,1728)$. Using Algorithm 7.10, we compute the following

integrals:

$$\int_b^P \omega_0\omega_1 = 4\cdot 11 + 4\cdot 11^2 + 7\cdot 11^3 + 9\cdot 11^4 + 5\cdot 11^6 + O(11^7)$$
$$\int_b^P \omega_0 = 4\cdot 11 + 7\cdot 11^2 + 9\cdot 11^3 + 3\cdot 11^4 + 5\cdot 11^5 + 7\cdot 11^6 + O(11^7)$$
$$\int_b^Q \omega_0\omega_1 = 4\cdot 11 + 4\cdot 11^2 + 7\cdot 11^3 + 9\cdot 11^4 + 5\cdot 11^6 + O(11^7)$$
$$\int_b^Q \omega_0 = 7\cdot 11 + 3\cdot 11^2 + 11^3 + 7\cdot 11^4 + 5\cdot 11^5 + 3\cdot 11^6 + O(11^7)$$
$$\int_b^R \omega_0\omega_1 = 5\cdot 11 + 6\cdot 11^2 + 7\cdot 11^3 + 5\cdot 11^4 + 3\cdot 11^5 + 9\cdot 11^6 + O(11^7)$$
$$\int_b^R \omega_0 = 3\cdot 11 + 7\cdot 11^2 + 2\cdot 11^3 + 3\cdot 11^4 + 7\cdot 11^6 + O(11^7),$$

and we see that the ratio in Corollary 8.2 is constant on integral points:

$$\frac{D_2(P)}{(\log_{\omega_0}(P))^2} = \frac{D_2(Q)}{(\log_{\omega_0}(Q))^2} = \frac{D_2(R)}{(\log_{\omega_0}(R))^2} = 3\cdot 11^{-1} + 6 + 2\cdot 11 + 10\cdot 11^2 + 3\cdot 11^3 + 5\cdot 11^4 + O(11^5).$$

However, for $S = (103, 980)$, which is not integral on $\mathcal{E}$, we see that

$$\int_b^S \omega_0\omega_1 = 3\cdot 11 + 10\cdot 11^2 + 4\cdot 11^3 + 10\cdot 11^4 + 7\cdot 11^5 + 10\cdot 11^6 + O(11^7)$$
$$\int_b^S \omega_0 = 11 + 7\cdot 11^3 + 5\cdot 11^5 + O(11^7)$$
$$\frac{D_2(S)}{(\log_{\omega_0}(S))^2} = 3\cdot 11^{-1} + 10 + 6\cdot 11 + 9\cdot 11^2 + 8\cdot 11^3 + 6\cdot 11^4 + O(11^5).$$

Example 8.4. We give a variation on Example 4 in [3]. Let E be the rank 1 elliptic curve $y^2 = x^3 - 16x + 16$, with minimal model $\mathcal{E}$ having Cremona label '37a1'. Letting P, Q be two fixed integral points on E, we can use the Link lemma to rewrite Theorem 8.1 so that the relevant double integral is no longer from a tangential basepoint. Indeed, integral points z occur in the zero set of

$$\left(\left(\int_b^z \omega_0\right)^2 - \left(\int_b^P \omega_0\right)^2\right) \frac{\int_P^Q \omega_0\omega_1 + \int_P^Q \omega_0 \int_b^P \omega_1}{(\int_b^Q \omega_0)^2 - (\int_b^P \omega_0)^2} - \left(\int_P^z \omega_0\omega_1 + \int_P^z \omega_0 \int_b^P \omega_1\right)$$

Slightly modifying Algorithm 7.4 to take as endpoint a parameter z (see [1, §7.2.2] for more details), we can recover the integral points

$$\{(0, \pm 4), (4, \pm 4), (-4, \pm 4), (8, \pm 20), (24, \pm 116)\}.$$

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