Enumeration of totally real number fields of bounded root discriminant

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$$B < 4\pi e^{1+\gamma} < 60.840$$

(or $B < 8\pi e^{\gamma+\pi/2} < 215.333$ on the GRH), we have $NF(n, B) = \emptyset$ for n sufficiently large and so the set $NF(B)$ is finite.

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n	$\#NF(n, 14)$	Prim F	Imprim F	Min d_F	Min δ_F
2	59	59	0	5	2.236
3	86	86	0	49	3.659
4	277	117	160	725	5.189
5	170	170	0	14641	6.809
6	263	104	159	300125	8.182
7	301	301	0	20134393	11.051
8	62	19	43	282300416	11.385
9	11	6	5	9685993193	12.869
10	0	0	0	443952558373?	14.613?
Total	1229	862	367	-	-

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Conjecture

Let $F = \mathbb{Q}(\alpha)$ *where* α *is a root of the polynomial*

$$x^{10} - 11x^8 - 3x^7 + 37x^6 + 14x^5 - 48x^4 - 22x^3 + 20x^2 + 12x + 1.$$

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The number field F (though not this polynomial) already appears in the tables of Klüners-Malle. It is a quadratic extension of the second smallest totally real quintic field, of discriminant 24217 (an S_5 extension).

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One may place alternative constraints on the signature of the fields F under consideration or even analogous p-adic conditions. However, totally real fields are interesting for many reasons.

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Totally real octics of moderate discriminant are good candidates as the base field for such a tower (coming from the Golod-Shafarevich bound). In joint work with Martin, we are now searching for a better tower.

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We also recently enumerated all CM-extensions K/F with *higher relative class number* at most sixteen, generalizing the Gauss class number 1 problem to higher K-groups.

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Results and timings

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n	2	3	4	5	6	7	8	9	10
$\Delta(n)$	30	25	20	17	16	15.5	15	14.5	14
f	443	4922	57721	244600	3242209	$1.7 \cdot 10^7$	$1.2 \cdot 10^8$	$9.5 \cdot 10^8$	$2.5 \cdot 10^9$
F	273	630	1273	674	802	301	164	15	0
CPU time	0.2s	2.2s	26.8s	1m25s	17m3s	2h59m	1d4.5h	17d21h	193d
Imprim f	0	0	7059	0	62532	0	239404	15658	945866
Imprim F	0	0	702	0	420	0	100	6	0
CPU time	-	-	4m22s	-	8m38s	-	1h56m	16m53s	11h27m
Total fields	273	630	1578	674	827	301	164	15	0

The CPU time is relative to the processor of a desktop machine (Opteron 1.8GHz, Athlon Dual Core 2.0GHz, and Celeron 2.53GHz).

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for an explicit bound $C(n, B)$.

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(We must also deal with the possibility that F is imprimitive and $\mathbb{Q}(\alpha) \neq F$.)

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$$\#NF(n, B) = O\left(B^{n \exp(C\sqrt{\log n})}\right)$$

for some absolute constant C (Ellenberg-Venkatesh).

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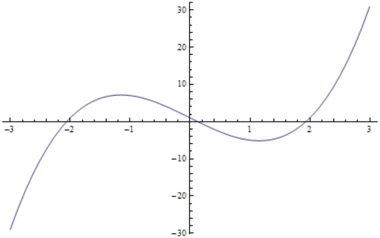

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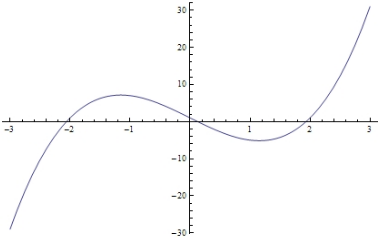

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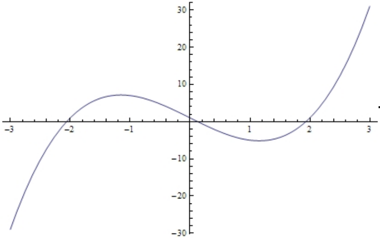

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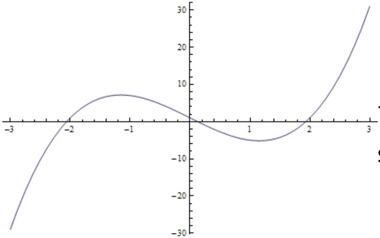

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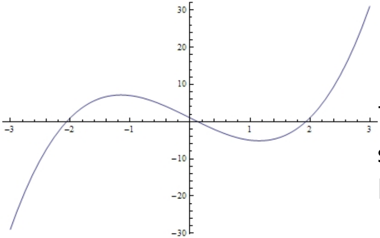

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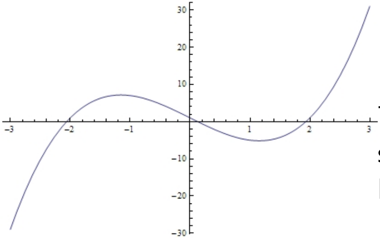

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In a similar way, using Lagrange multipliers (Pohst) we find a bound on the largest β_3 and smallest root β_0 of $f(x)$ which yields $f_3(\beta_3) = g_3(\beta_3) + a_{n-3} > 0$ so $-g_3(\beta_3) < a_{n-3} < -g_3(\beta_0)$.

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Thanks!