On the Diophantine Equation $x^2 + 2^\alpha 5^\beta 13^\gamma = y^n$

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Background and history

The history of the Diophantine equation

$$x^2 + C = y^n, \quad x \geq 1, \quad y \geq 1, \quad n \geq 3,$$

is very rich. This equation is called *Lebesque-Nagell* equation.

• In 1850, Lebesgue was the first to study this equation when $C = 1$. He found no solutions.

• In 1923, Nagell studied the equation for $C = 3, 5$. He also found no solutions.

• In 1943, Ljunggren extended a result of Fermat to prove that the equation for $C = 2$ has only the solution $x = 5, y = 3$.

• A particular case is the *Ramanujan-Nagell* equation $x^2 + 7 = 2^n$ solved by Nagell in 1948.

• The case $C = -1$ was solved by Chao Ko in 1965. The only solution is $x = 3,\ y = 2$. This is a special case of the Catalan equation.

• J.H.E. Cohn solved the above equation for several values of the parameter C in the range $1 \leq C \leq 100$ in 1992 and 1993.

• Mignotte and De Weger studied the cases $C = 74, 86$ in 1996.

• Bennett and Skinner applied the modular approach to solve the equation for $C = 55, 95$ in 2004.

• In 2006, Bugeaud-Mignotte-Siksek used classical and modular approaches to solve the rest of the cases, for $1 \leq C \leq 100$.

Recently, several authors become interested in the case $C = p^k$, where p is a prime number or C is the product of prime powers.

• The case $p = 2$ was studied by Arif-Abu Muriefah (1997) and Le (2002).

• The case $p = 3$ was solved by Arif-Abu Muriefah (1998), and Luca (2000).

• In 2000, Arif-Abu Muriefah studied the case $p = 5$ and k odd.

• Partial results for a general prime p appear in Arif-Muriefah (2002) and Le (2001).

• Luca - T. obtained a result when $C = 7^{2a}$.

• All the solutions when x and y are coprime and

$$C = 2^a \cdot 3^b$$

were found by Luca in 2002.

• The cases

$$C = 2^a \cdot 5^b \text{ and } C = 2^a \cdot 13^b$$

were studied by Luca and T.

• Abu Muriefah, Luca, and T. solved the equation when

$$C = 5^a \cdot 13^b.$$

• The first result when C is a product of higher number of factors was obtained by Pink in 2007. He considered

$$C = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7^\delta.$$

The main result

Theorem

The equation

$$x^2 + 2^\alpha 5^\beta 13^\gamma = y^n, \tag{0.1}$$

with $x \geq 1,\ y \geq 1,\ \gcd(x, y) = 1,\ n \geq 3,$
$\alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0;$
has no solution except for:
$n = 3$ *the solutions given in Table 1;*
$n = 4$ *the solutions given in Table 2;*
$n = 5$ $(x, y, \alpha, \beta, \gamma) = (401, 11, 1, 3, 0);$
$n = 6$ $(x, y, \alpha, \beta, \gamma) \in$
$\{(25, 3, 3, 0, 1), (23, 3, 3, 2, 0), (333, 7, 3, 1, 2), (521, 9, 5, 4, 1)\};$
$n = 7$ $(x, y, \alpha, \beta, \gamma) = (43, 3, 1, 0, 2);$
$n = 8$ $(x, y, \alpha, \beta, \gamma) \in \{(79, 3, 6, 1, 0), (49, 3, 6, 1, 1)\};$
$n = 12$ $(x, y, \alpha, \beta, \gamma) = (521, 3, 5, 4, 1).$

For $n \geq 3$ fixed, let $(x, y, \alpha, \beta, \gamma)$ be a solution to the equation.

α	β	γ	x	y	α	β	γ	x	y
0	0	1	70	17	8	0	2	6183	337
0	2	2	142	29	14	2	10	137411503	422369
6	2	2	98233	2129	2	4	1	441	61
1	0	0	5	3	3	0	1	25	9
7	6	0	383	129	9	12	1	1071407	14049
1	0	1	1	3	9	0	3	181	105
1	0	1	207	35	9	1	6	83149	2681
7	0	1	57	17	3	1	2	333	49
7	0	1	18719	705	9	2	0	17771	681
1	6	1	8553	419	3	2	0	23	9
25	0	1	15735	881	9	2	1	109513	2289
1	2	3	151	51	15	4	3	11706059	51561
13	2	5	1075281	10721	4	0	2	47	17
7	3	2	3114983	21329	4	4	6	1397349	12601
1	4	0	9	11	5	1	2	3017	209
1	4	2	9823	459	5	2	0	261	41
1	16	2	46679827	130659	11	2	1	1217	129
2	0	0	11	5	5	2	3	103251	2201
2	0	2	27045	901	5	4	1	521	81

Table: Solutions for $n = 3$

α	β	γ	x	y
4	1	0	1	3
0	1	1	4	3
12	1	1	959	33
5	0	0	7	3
5	4	1	521	27
5	2	1	2599	51
6	1	0	79	9
6	1	1	49	9
10	1	1	16639	129
7	2	1	391	21

Table: Solutions for $n = 4$.

Ideas of the proof: the case $n = 3, 6, 12$

Lemma

- *When* $n = 3$*, then the only solutions to equation* (0.1) *are given in Table 1.*

- *For* $n = 6$*, the only solutions are*

$$(25, 3, 3, 0, 1), (23, 3, 3, 2, 0), (333, 7, 3, 1, 2), (521, 9, 5, 4, 1);$$

- *For* $n = 12$ *then* $(521, 3, 5, 4, 1)$ *is the only solution.*

Factoring the initial equation leads to the equations

$$\left(\frac{x}{z^3}\right)^2 + A = \left(\frac{y}{z^2}\right)^3, \tag{0.2}$$

where A is implicitly defined by

$$2^{\alpha}\, 5^{\beta}\, 13^{\gamma} = Az^6.$$

One can see that $A = 2^{\alpha_1}\, 5^{\beta_1}\, 13^{\gamma_1}$
with $\alpha_1,\ \beta_1,\ \gamma_1 \in \{0, 1, 2, 3, 4, 5\}$. We thus get

$$V^2 = U^3 - 2^{\alpha_1} \cdot 5^{\beta_1} \cdot 13^{\gamma_1}, \tag{0.3}$$

with $U = y/z^2,\ V = x/z^3$ and $\alpha_1,\ \beta_1,\ \gamma_1 \in \{0, 1, 2, 3, 4, 5\}$.

We used Magma to find the solutions.

Looking in the list of solutions those whose y is a perfect power, we determine the solutions for $n = 6, 12$.

Ideas of the proof: the case $n = 4,\ 8$

Lemma

• *If* $n = 4$*, then the only solutions to equation* (0.1) *are given in Table 2.*

• *If* $n = 8$*, then the only solutions to equation* (0.1) *are* $(79, 3, 6, 1, 0), (49, 3, 6, 1, 1)$*.*

Equation (0.1) can be written as

$$\left(\frac{x}{z^2}\right)^2 + A = \left(\frac{y}{z}\right)^4, \tag{0.4}$$

where A is fourth-power free and defined implicitly by

$$2^{\alpha}\, 5^{\beta}\, 13^{\gamma} = Az^4.$$

One can see that $A = 2^{\alpha_1}\, 5^{\beta_1}\, 13^{\gamma_1}$
with $\alpha_1,\ \beta_1,\ \gamma_1 \in \{0, 1, 2, 3\}$.

Hence, the problem consists in determining the $\{2, 5, 13\}$-integer points on the totality of the 64 elliptic curves

$$V^2 = U^4 - 2^{\alpha_1}\, 5^{\beta_1}\, 13^{\gamma_1}, \tag{0.5}$$

with $U = y/z,\ V = x/z^2$ and $\alpha_1,\ \beta_1,\ \gamma_1 \in \{0, 1, 2, 3\}$.
Magma does the job.

For $n = 8$, we look in the list of solutions those whose y is a perfect square.

Ideas of the proof: the case $n \geq 5$ and prime

Lemma

The Diophantine equation (0.1) *has no solution with* $n \geq 5$ *prime except for*

- $n = 5 \quad (x, y, \alpha, \beta, \gamma) = (401, 11, 1, 3, 0)$;
- $n = 7 \quad (x, y, \alpha, \beta, \gamma) = (43, 3, 1, 0, 2)$.

Let $\mathbb{K} = \mathbb{Q}[i\sqrt{d}]$, where $C = 2^\alpha\, 5^\beta\, 13^\gamma = dz^2$,

$$d = 1, 2, 5, 10, 13, 26, 65, 130$$

according to the parities of the exponents α, β, and γ.

We factor the above equation in $\mathbb{K}$ to get

$$\left(x + i\sqrt{d}\, z\right)\left(x - i\sqrt{d}\, z\right) = y^p. \tag{0.6}$$

Assume that

$$x + i\sqrt{d}z = \eta^p \tag{0.7}$$

holds with some algebraic integer $\eta \in \mathcal{O}_{\mathbb{K}}$.

We obtain

$$2i\sqrt{d}\, 2^a\, 5^b\, 13^c = \eta^p - \bar{\eta}^p. \tag{0.8}$$

If we set $\eta = u + v\sqrt{d}$, then we deduce that $v \mid 2^a\,5^b\,13^c$, and that

$$\frac{2^a\,5^b\,13^c}{v} = \frac{\eta^p - \bar{\eta}^p}{\eta - \bar{\eta}} \in \mathbb{Z}. \qquad (0.9)$$

Let $\{L_m\}_{m\geq 0}$ be the *Lucas sequence* of general term $L_m = \dfrac{\eta^m - \bar{\eta}^m}{\eta - \bar{\eta}}$, for all $n \geq 0$.

Equation (0.9) leads to the conclusion that

$$P(L_p) = P\left(\frac{2^a 5^b 13^c}{v}\right). \qquad (0.10)$$

Using the Primitive Divisor Theorem for Lucas sequences implies that if $p \geq 5$, then L_p has a *primitive* prime factor except for finitely many pairs $(\eta, \bar{\eta})$ and all of them appear in tables obtained by Bilu-Hanrot-Voutier and Abouzaid.

We obtain that $d \in \{2, 5, 10\}$ and $p = 5, 7$.

Further work:

- Can one obtain similar results the equation

$$a_1 x^2 + C = a_2 y^n?$$

- For $a_2 = 2$, some results are obtained by Ljunggren, Bugeaud, Tengely, Muriefah-Luca-Siksek-Tengely,...

- For $a_2 = 4$, some results are obtained by Bugeaud, Mignotte, Maohua, Bilu, Yuan, Arif-Al Ali,...

- For $a_2 = 3$, ???... It may be a little difficult. But something can be done.