Faster Multiplication in $\mathbb{F}_2[X]$

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Post-doc position

The CACAO project (Nancy, France) offers a post-doc position for working on the Number Field Sieve ; in particular:

- implementation concerns.
- Software speed-up.
- Large scale distribution.

This is part an ongoing project on NFS, named CADO:

`http://cado.gforge.inria.fr/`

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Plan

1. **Introduction**
2. **Small sizes**
3. **Medium sizes**
4. **Large sizes**

1. **Introduction**
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Why ?

We focus on polynomial multiplication over $\mathbb{F}_2[x]$.

This is used in many contexts:

- polynomial factorization, irreducibility tests ;
- (some) crypto applications ;
- less obvious: sparse linear algebra over $\mathbb{F}_2$;
- and more.

How does data look like ?

Binary polynomial $x^3 + x^2 + 1 \quad \rightarrow \quad$ machine integer$(1101)_2$ (“dense”).

- up to degree 63: one machine word (64-bit).
- degree 64 to 127: two words.
- ...

In hardware:

- `add` is trivial ;
- `mul` is easy ; much easier than integer `mul`.
- Not our business.

In software:

- `add` is trivial (`xor`) ;
- `mul` is tedious (no PCMULQDQ yet !).

What do we do ?

We are interested in:

- software.
- speed everywhere: from 64 to 2^{32} coefficients (think recursion).

Existing software

Existing software typically has:

- Possibly fast multiplication for $1, 2 \ldots$ up to a few words.
- Karatsuba multiplication above.

Main reference: Victor Shoup's NTL: `shoup.net/ntl`

Very rarely (if ever), one finds:

- Code that takes advantage of CPU-specific instructions ;
- Toom-Cook multiplication ;
- Fast multiplication for unbalanced operands ;
- FFT (Schönhage ternary + Cantor additive).

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Below degree 64: `mul1`

Classical: $c = a \times b$ computed with a (fixed-) window method.

- Tabulate multiples $g \times b$, for $\deg g < s$ ($s =$ window size).
- Split $a = A_0 + A_1 x^s + A_2 x^{2s} + \cdots$.
- Accumulate $c = A_0 \times b + (A_1 \times b)x^s + (A_2 \times b)x^{2s} + \cdots$.

Operations required: shifts, XORs.

For degree below 64, we work with machine words only.

- We measure the best window size with experiments.
- 64×64: ~ 75 Intel `core2` cycles ; ~ 85 AMD `k8` cycles.
- $64k \times 64$ would work the same way.

Using SIMD capabilities

What about 128×128 ?

- Karatsuba $\Rightarrow$ three 64×64.
- Schoolbook requires $a \times b_{\text{low}}$ and $a \times b_{\text{high}}$ $\Rightarrow$ two 128×64.
- BUT $a \times b_{\text{low}}$ and $a \times b_{\text{high}}$ can be computed in a SIMD-manner.
- SIMD instructions on `x86_64` provide the necessary shifts and XORs.
  - Accessible with compiler builtins (`gcc`, `icc`, MSVC).
  - Assembly is not absolutely necessary.

128×128:

- ~ 129 Intel `core2` cycles ;
- ~ 226 AMD `k8` cycles.
- Faster than Karatsuba here.

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Medium sizes

Classical: from 2 to 9 machine words, hard-code Karatsuba multiplication.
$\Rightarrow$ No branching.

Example for `mul4`:

```
mul2 (c, a, b);
mul2 (c + 4, a + 2, b + 2);
aa[0] = a[0] ^ a[2]; aa[1] = a[1] ^ a[3];
bb[0] = b[0] ^ b[2]; bb[1] = b[1] ^ b[3];
c24 = c[2] ^ c[4];
c35 = c[3] ^ c[5];
mul2 (ab, aa, bb);
c[2] = ab[0] ^ c[0] ^ c24; c[3] = ab[1] ^ c[1] ^ c35;
c[4] = ab[2] ^ c[6] ^ c24; c[5] = ab[3] ^ c[7] ^ c35;
```

Medium sizes

Classical: from 2 to 9 machine words, hard-code Karatsuba multiplication.

$\Rightarrow$ No branching.

Cycle counts, Intel `core2`.

$\deg$	NTL	LIDIA	ZEN	this paper
63	99	117	158	**75**
127	368	317	480	**132**
191	703	787	1 005	**364**
255	1 130	988	1 703	**410**
319	1 787	1 926	2 629	**806**
383	2 182	2 416	3 677	**850**
447	3 070	2 849	4 960	**1 242**
511	3 517	3 019	6 433	**1 287**

What comes next ?

Toom-3: $\deg a < 3k$, write $a = A(x, x^k)$, $A(x,t) = a_0(x) + a_1(x)t + a_2(x)t^2$.

- Evaluate $(A(x, x_i))_{i=0,1,2,3,4}$ and $(B(x, x_i))_{i=0,1,2,3,4}$
- Multiply: $C(x, x_i) = A(x, x_i)B(x, x_i)$.
- Interpolate: recover $C(x,t)$ from $(C(x, x_i))_{i=0,1,2,3,4}$

Misbelief:
- This is only for $\#K \geq 4 \ldots$
- because we need 5 evaluation points (in $\mathbb{P}^1(K)$).

- We can use: $0, 1, \infty, x, x^{-1}$.
- Often better: $0, 1, \infty, x^{64}, x^{-64}$: avoids shifts.
- The degrees in recursive calls increase mildly.

See paper for timings.

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Large sizes

We are also interested in multiplication in the FFT range.

Several options:

- integer FFT and (huge) padding (Krönecker-Schönhage).
- Cantor’s additive FFT algorithm.
- Schönhage’s ternary FFT algorithm.

Cantor's additive FFT

Assume we are given fast polynomial multiplication in $F_k = \mathbb{F}_{2^{2^k}} = \mathbb{F}_2[\gamma]$.

We use it for multiplication in $\mathbb{F}_2[x]$.

- Separate coefficients of a and b in blocks of 2^{k-1} coefficients:
  - Write $a = A(x, x^{2^{k-1}})$, $A(x,t) = a_0(x) + a_1(x)t + a_2(x)t^2 + \cdots$.
  - $\tilde{a} = A(\gamma, t) \in F_k[t]$.
- $\tilde{a} \times \tilde{b}$ has coefficients in F_k.
- Since $\deg a_i b_j < 2^k$, then $c = a \times b$ is such that $\tilde{c} = \tilde{a} \times \tilde{b}$.

Multiplying in F_k

Let $s_1(x) = x^2 + x$, and $s_i(x) = \underbrace{s_1(s_1(\cdots s_1}_{i \text{ times}}(x)\cdots))$.

s_i satisfies many properties:

- s_i is sparse ; s_i is linear ; $s_{2^k} = x^{2^{2^k}} + x$.
- Let $2^k \geq i$ and $W_i = \left\{\alpha \in \mathbb{F}_{2^{2^k}} \mid s_i(\alpha) = 0\right\} = \operatorname{Ker} s_i$.
  W_i is a sub-vector space of $\mathbb{F}_{2^{2^k}}$; $\dim W_i = i$.

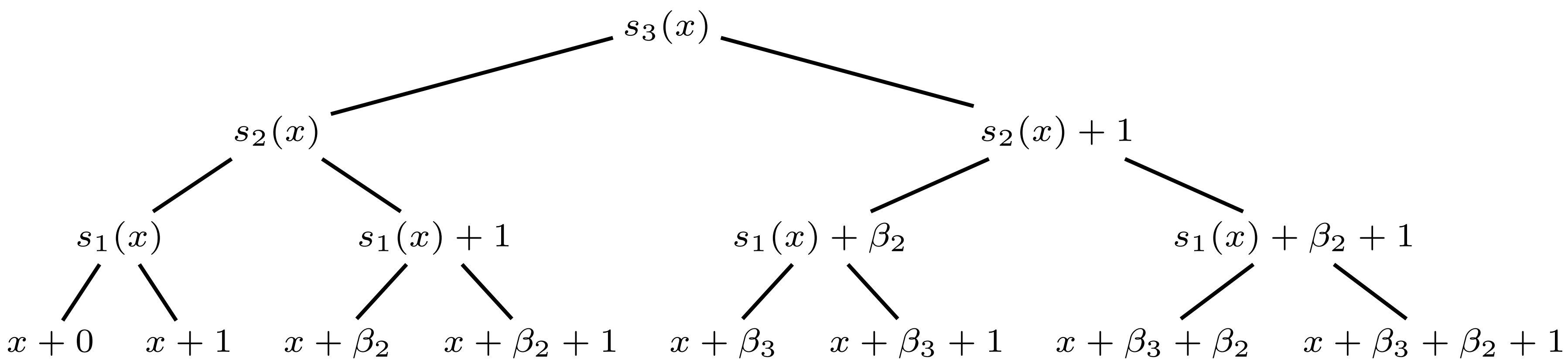
How do we multiply $h = f \times g$ in $F_k[x]$?

- Evaluate f and g at points of some W_i.
- multiply pointwise to obtain $\{f(\alpha) \times g(\alpha),\ \alpha \in W_i\}$.
- Interpolate: recover h from $\{f(\alpha) \times g(\alpha),\ \alpha \in W_i\}$.

Multiplying in F_k (2)

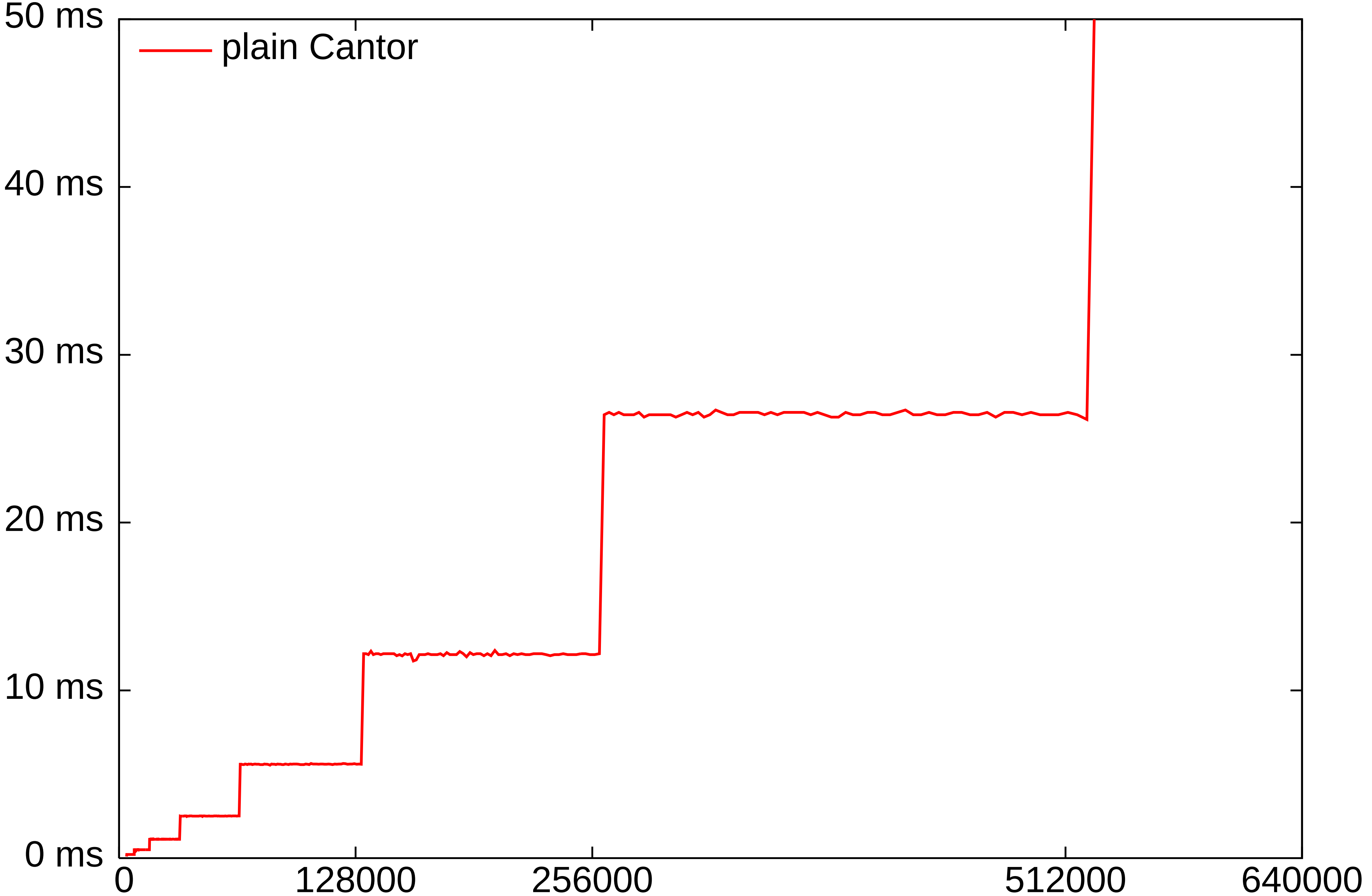
Multi-evaluating at W_i is done with a sub-product tree:

$$\{f(\alpha),\ \alpha \in W_i\} = \{f \bmod (x+\alpha),\ \alpha \in W_i\}\,.$$

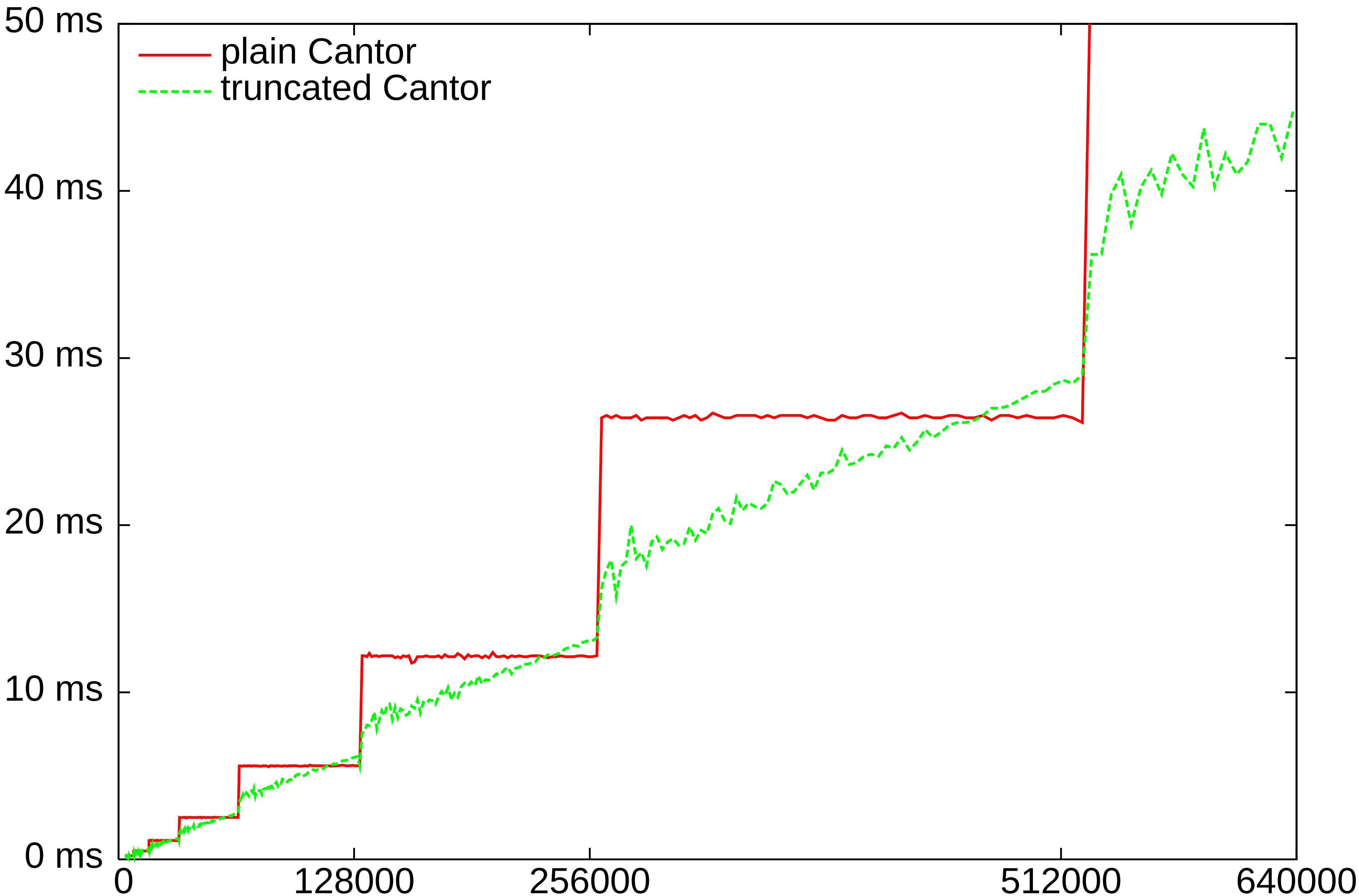


- right-child $= 1 +$ left-child.
- Only the constant coefficients are in extension fields.
- s_j is sparse, so reduction is cheap.

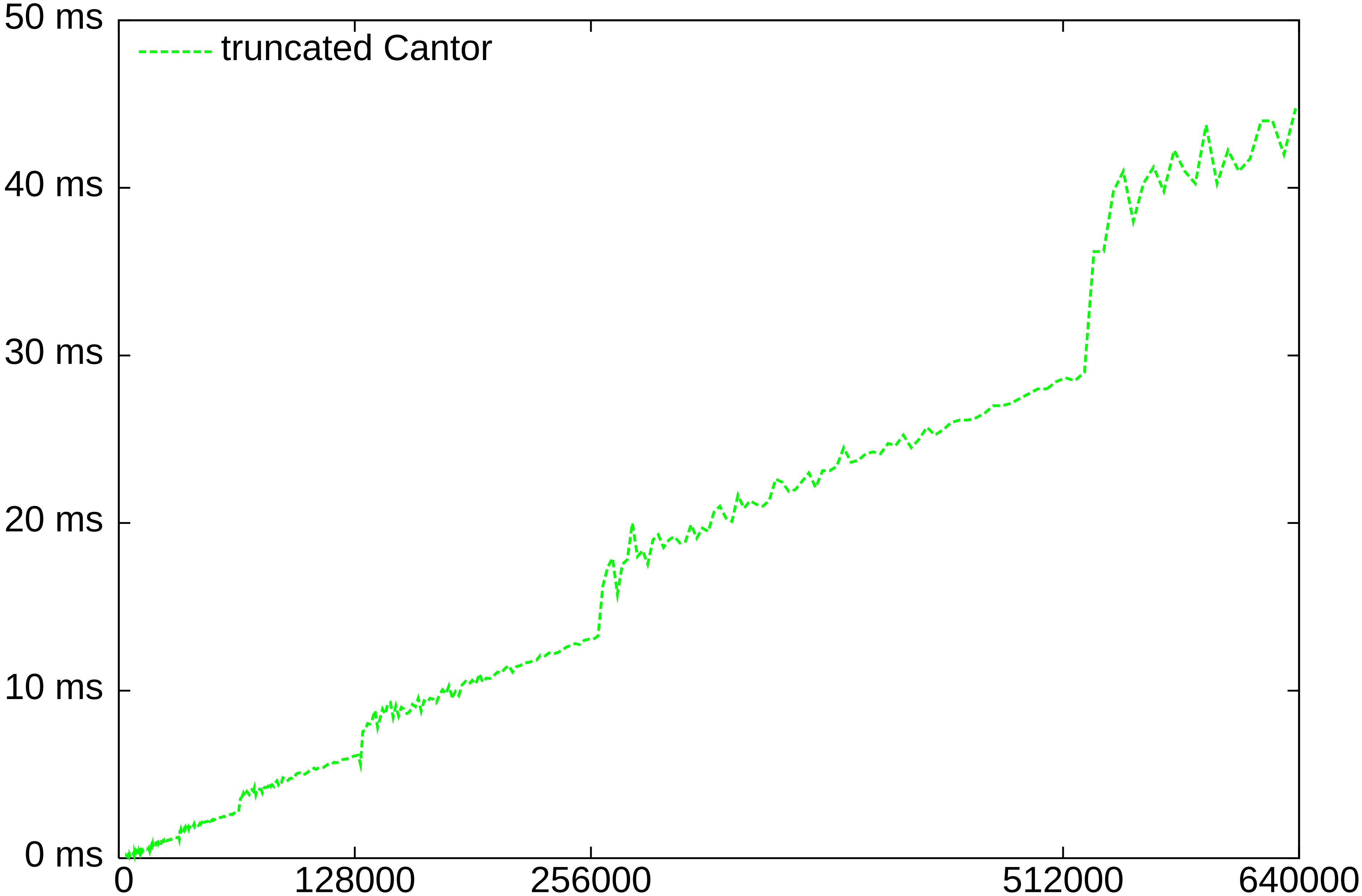
Performance of additive FFT



Performance of additive FFT



Performance of additive FFT



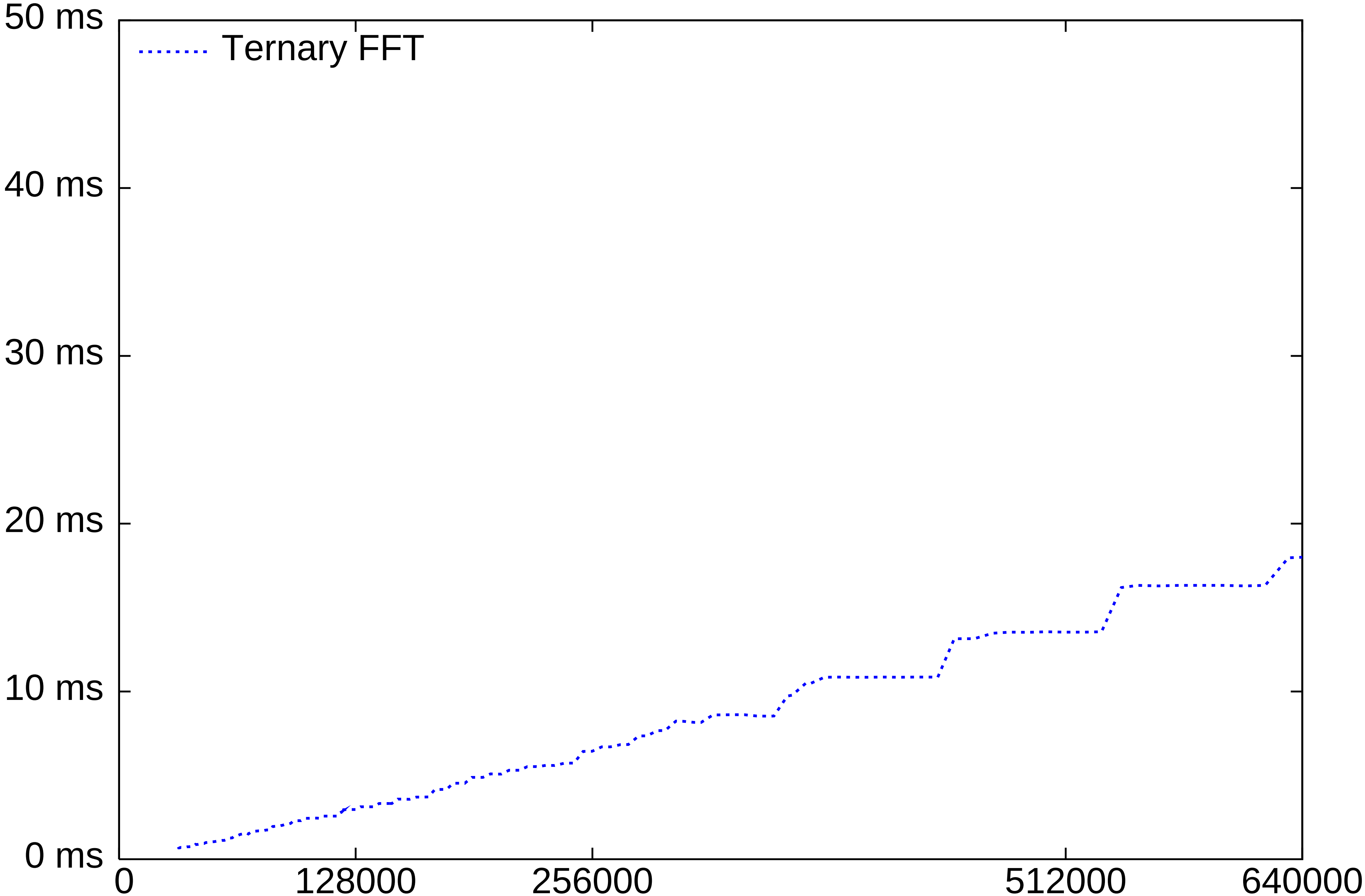
Schönhage's ternary FFT algorithm

- FFT typically calls for 2^n roots of unity ; bad for $\mathrm{char}\, K = 2$.
- Schönhage (1977): work in $R = \mathbb{F}_2[x]/x^{2L} + x^L + 1$, where $L = \lambda 3^{k-1}$.
- x^λ is a 3^k-th root of 1 in R.
- Use ternary FFT to multiply polynomials of degree $< 3^k$ in $R[t]$.
  - Evaluate $\hat{f} = \left\{ f(x^{\lambda i}),\ 0 \le i < 3^k \right\}$. (same for $\hat{g}$).
  - Multiply pointwise to obtain $\widehat{fg}$; multiplications in R: recurse.
  - Interpolate to recover fg ; FFT again since $\hat{\hat{f}} = f$.

Same clumping technique as before $\Rightarrow$ multiplication in $\mathbb{F}_2[x]$.

- In effect, we multiply modulo $x^N + 1$ (for some $N > \deg ab$).
- See details in paper.

Schönhage FFT



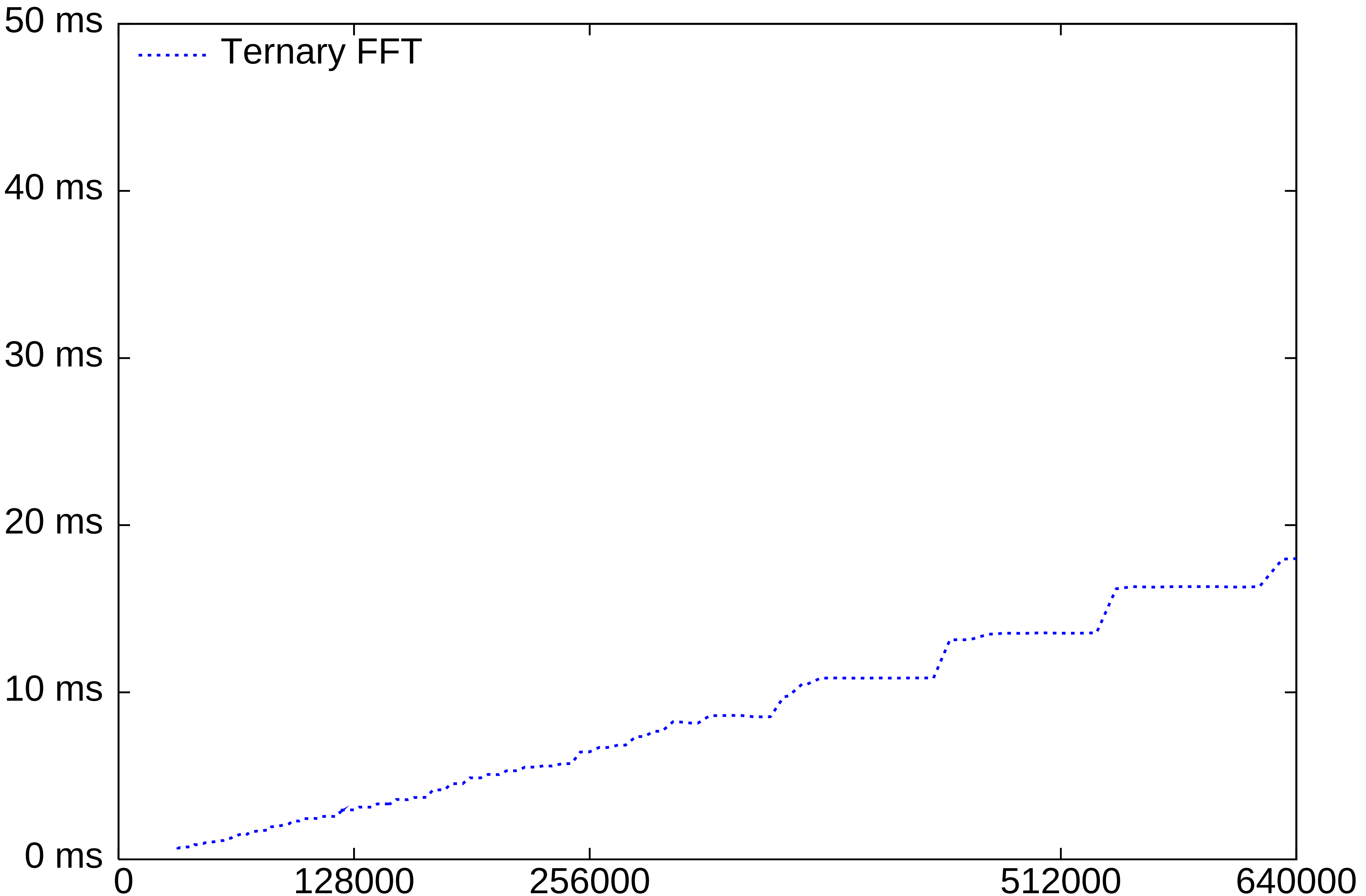
Splitting the ternary FFT

There is a (mild) staircase effect.

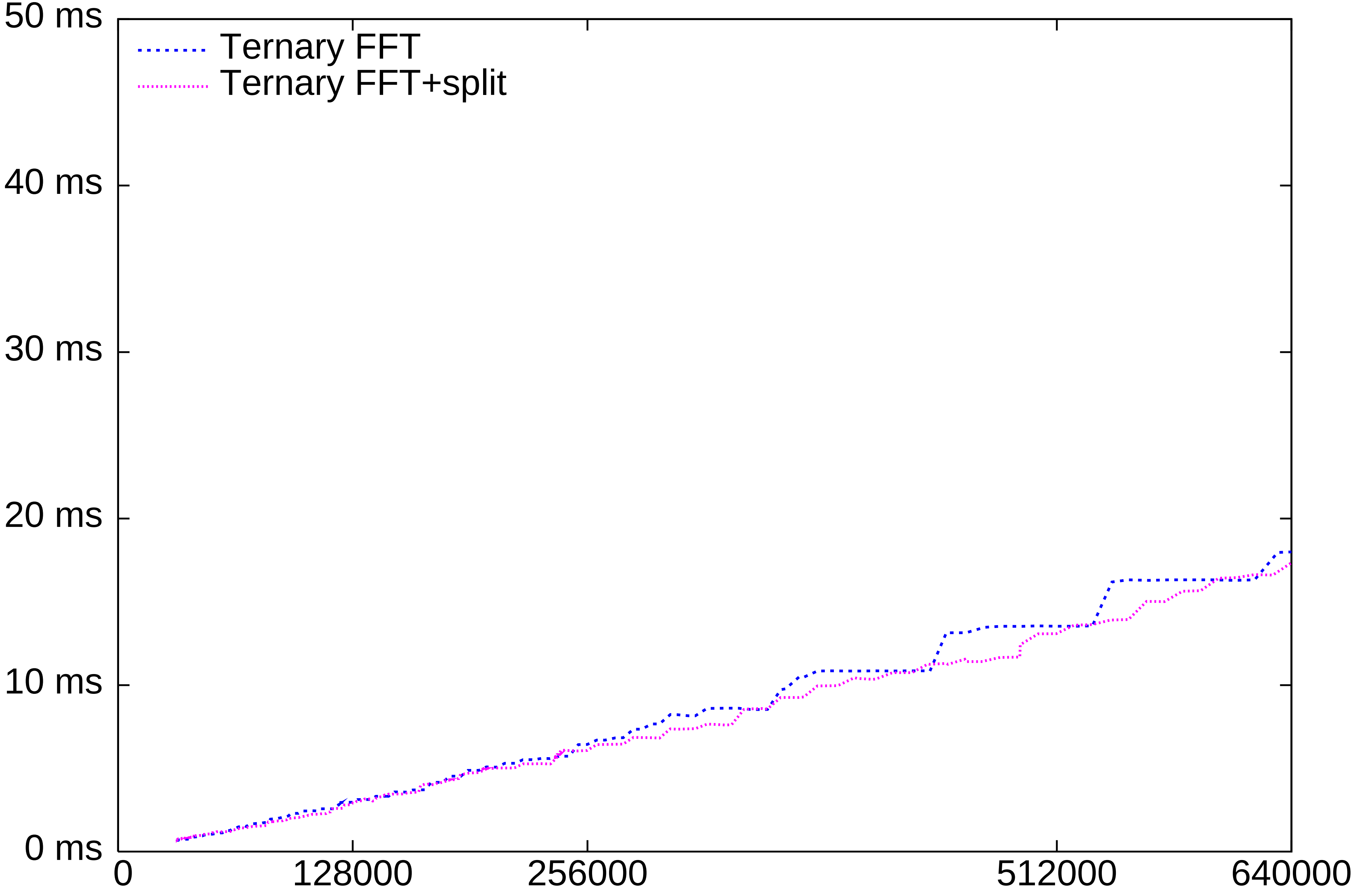
We can compute a product of degree $< N$ by splitting:

- Compute one product modulo $N' > N/2$.
- Compute another product modulo $N'' > N'$.
- Very simple XORs do the reconstruction.

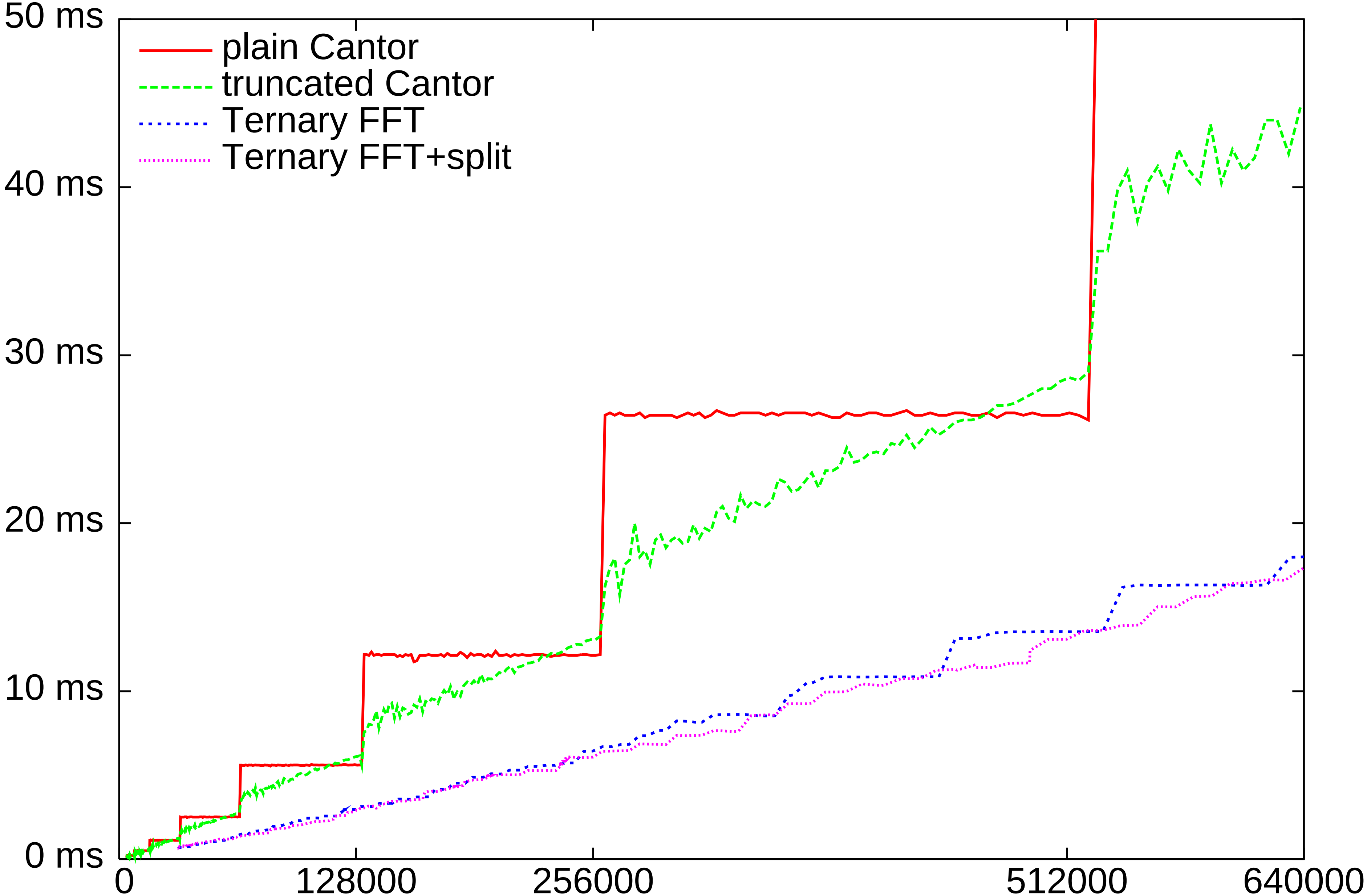
Schönhage FFT + splitting



Schönhage FFT + splitting



Comparison Cantor – Schönhage



Comparison Cantor – Schönhage

A word of caution:

- Additive FFT has cheap pointwise products.
- Ternary FFT has cheap evaluation / interpolation.

When transforms can be reused (matrices over $\mathbb{F}_2[x]$), additive FFT wins. Example for $\deg ab < 2^{20}$:

- Additive FFT: 57 ms, 2.3 ms in pointwise mults.
- Ternary FFT: 28 ms, 18 ms in pointwise mults.
- $n \times n$ matrix mult: $c_{\text{eval/interp}} * n^2 + c_{\text{pointwise}} * n^3$
- Additive FFT faster for 3×3 matrices and above.

Conclusion

- Significant speed-ups over existing software.
- Openly available implementation of two FFT algorithms.
- Accessible from `rpbrent.com/gf2x.html`