Functorial Properties of Stark Units in Multiquadratic Extensions

Jonathan W. Sands[1] and Brett A. Tangedal[2]

[1] University of Vermont, Burlington VT, 05401, USA
jwsands@uvm.edu

[2] University of North Carolina at Greensboro, Greensboro NC, 27402, USA
batanged@uncg.edu

May 18, 2008

UNCG

Let $F = \mathbb{Q}(\theta)$, where $\theta^2 = D$ and D is a square-free integer > 1. Define the first embedding

$$e_1 : F \hookrightarrow \mathbb{R} \quad \text{by sending} \quad \theta \mapsto \sqrt{D}$$

and the second embedding

$$e_2 : F \hookrightarrow \mathbb{R} \quad \text{by sending} \quad \theta \mapsto -\sqrt{D}.$$

For $\alpha \in F$, we use the notation $e_j(\alpha) = \alpha^{(j)}$.
Let

$d = d_F$ be the discriminant of F,

$h = h_F$ the class number of F,

$u = u_F$ the unique fundamental unit of F with $1 < u^{(1)}$, and

$\mathcal{O}_F$ the integral closure of $\mathbb{Z}$ in F.

The Kronecker symbol $\chi_d(n)$, $n \in \mathbb{Z}^+$, attached to F is defined as follows. For any given rational prime p,

$$\chi_d(p) = \begin{cases} 1 & \text{if} \quad p \quad \text{splits in } \mathcal{O}_F, \\ -1 & \text{if} \quad p \quad \text{is inert in } \mathcal{O}_F, \\ 0 & \text{if} \quad p \quad \text{ramifies in } \mathcal{O}_F\text{, i.e., iff } p \mid d. \end{cases}$$

We set $\chi_d(1) = 1$ and extend χ_d to all other positive integers multiplicatively. Using quadratic reciprocity, one may show that χ_d defines a primitive (even) Dirichlet character of conductor d. The corresponding L–function

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}$$

is absolutely convergent for $\Re(s) > 1$ and may be analytically continued to an entire function.

The value $L(1, \chi_d)$ was computed expicitly by Dirichlet and he found that

$$L(1, \chi_d) = \frac{2h \log u^{(1)}}{\sqrt{d}}.$$

The function $L(s, \chi_d)$ has a first order zero at $s = 0$ and the functional equation for $L(s, \chi_d)$ relates the value $L(1, \chi_d)$ to the leading coefficient $L'(0, \chi_d)$ at $s = 0$ giving the relation

$$L'(0, \chi_d) = h \log u^{(1)},$$

which is cleaner than the value at $s = 1$.

If χ_0 is the trivial character modulo d, we define

$$L(s,\chi_0) = \sum_{\substack{(n,d)=1 \\ n \geq 1}} \frac{1}{n^s} = \zeta(s) \prod_{p \mid d} (1 - p^{-s}).$$

If $t = \#$ of distinct prime divisors of d, then $L(s,\chi_0)$ has a zero of order t at $s = 0$ upon meromorphic continuation to the whole complex plane. We have $t = 1$ precisely when $D = 2$ or $D = p$, where p is a prime $\equiv 1 \pmod 4$. We have

$$L'(0,\chi_0) = \begin{cases} -\frac{1}{2}\log D & \text{if} \quad t = 1, \\ 0 & \text{if} \quad t \geq 2. \end{cases}$$

Let $G = \text{Gal}(F/\mathbb{Q}) = \{\sigma_0, \sigma\}$ and define χ_d on G by $\chi_d(p) = \chi_d(\sigma_p) \ \ \forall p \nmid d$ or, equivalently, $\chi_d(\sigma_0) = 1$ and $\chi_d(\sigma) = -1$. Similarly, $\chi_0(\sigma_0) = \chi_0(\sigma) = 1$.

Stark's Conjecture for the extension $F/\mathbb{Q}$ says that there exists an algebraic integer $\varepsilon \in \mathcal{O}_F$ such that

$$L'(0,\chi_0) = -\frac{1}{2}\sum_{\rho\in G}\chi_0(\rho)\log|\rho(\varepsilon)^{(1)}| = -\frac{1}{2}\log|N(\varepsilon)|, \qquad (1)$$

where $N(\varepsilon) = N_{F/\mathbb{Q}}(\varepsilon)$, and

$$L'(0,\chi_d) = -\frac{1}{2}\sum_{\rho\in G}\chi_d(\rho)\log|\rho(\varepsilon)^{(1)}| = -\frac{1}{2}\log\left|\frac{\varepsilon^{(1)}}{\sigma(\varepsilon)^{(1)}}\right|, \qquad (2)$$

and $F(\varepsilon^{1/2})$ is an abelian extension of $\mathbb{Q}$. This last "abelian condition" holds iff $N(\varepsilon)$ is a square in F. Therefore, we must have $N(\varepsilon) = D$ when $t = 1$ and $N(\varepsilon) = 1$ when $t \geq 2$ since we can't have $N(\varepsilon)$ negative and satisfy the abelian condition at the same time.

It is easy to check that

$$\varepsilon = u^{-h}\theta \quad \text{when} \quad t = 1 \; (\theta^2 = D)$$

and

$$\varepsilon = u^{-h} \quad \text{when} \quad t \geq 2$$

satisfy equations (1) and (2) and that $\varepsilon^{(1)} > 0$ in both cases. In order for $F(\varepsilon^{1/2})$ to be an abelian extension of $\mathbb{Q}$ we also require that $\sigma(\varepsilon)^{(1)} > 0$ as well. That this holds in all cases is nontrivial and is intimately related to the genus theory of quadratic fields. In the case where $t = 1$, genus theory says that $N(u) = -1$ and that h is odd. Therefore

$$\sigma(\varepsilon)^{(1)} = \left[\left(-\frac{1}{u}\right)^{-h} \cdot (-\theta)\right]^{(1)} = \left[u^h\theta\right]^{(1)} > 0,$$

and so $N(\varepsilon) = D$ is a square in F and the abelian condition is satisfied.

If $t \geq 2$ and $N(u) = 1$, then $\sigma(\varepsilon) = u^h$ and $\sigma(\varepsilon)^{(1)} > 0$. If $t \geq 2$ and $N(u) = -1$, then genus theory tells us that h is even. Therefore

$$\sigma(\varepsilon)^{(1)} = \left[\left(-\frac{1}{u}\right)^{-h}\right]^{(1)} = \left[u^h\right]^{(1)} > 0,$$

and so $N(\varepsilon) = 1$ is a square in F.

It happens often that the "Stark unit" ε is a square in the field in which it lies. Note that ε above is a square in F when h is even.

Let F be a fixed real quadratic field for the rest of the talk. We next consider relative quadratic extensions K/F and consider Stark's Conjecture in this new setting. Corresponding to K/F is a generalized Kronecker symbol χ defined just as before:
If $\mathfrak{p}$ is a prime ideal in $\mathcal{O}_F$, then

$$\chi(\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathfrak{p} \text{ splits in } \mathcal{O}_K, \\ -1 & \text{if } \mathfrak{p} \text{ is inert in } \mathcal{O}_K, \\ 0 & \text{if } \mathfrak{p} \text{ ramifies in } \mathcal{O}_K, \text{ i.e., iff } \mathfrak{p} \mid d(K/F). \end{cases}$$

We set $\chi((1)) = 1$ and extend χ to all other nonzero ideals in $\mathcal{O}_F$ multiplicatively. By class field theory, χ defines a ray class character, the finite part of whose conductor is equal to $d(K/F)$.

The corresponding L–function

$$L(s, \chi) = \sum \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s},$$

where the sum is taken over all nonzero ideals in $\mathcal{O}_F$, again defines an entire function upon analytic continuation. The order $r(\chi)$ of the zero of this function at $s = 0$ depends upon the signature of the field K in the following way:

$$r(\chi) = \begin{cases} 2 & \text{if} \quad K \quad \text{has signature } [4,0] \text{ (i.e., } K \text{ is totally real),} \\ 1 & \text{if} \quad K \quad \text{has signature } [2,1], \\ 0 & \text{if} \quad K \quad \text{has signature } [0,2] \text{ (i.e., } K \text{ is totally complex).} \end{cases}$$

The relative quadratic extensions of F of greatest interest to us are those with signature $[2, 1]$ and $r(\chi) = 1$.

If K has signature $[2, 1]$, then by class field theory there is at least one prime ideal from $\mathcal{O}_F$ that ramifies in $\mathcal{O}_K$. We define in this case

$$L(s, \chi_0) = \zeta_F(s) \cdot \prod_{\mathfrak{p} \mid d(K/F)} (1 - N\mathfrak{p}^{-s})$$

and note that $L'(0, \chi_0) = 0$. We may consider χ_0 and χ as characters on the Galois group $G = \mathrm{Gal}(K/F) = \{\sigma_0, \sigma\}$ just as before.

We will assume from now on that if K has signature $[2, 1]$ that the two real embeddings of K extend what we defined earlier as the first embedding of F into $\mathbb{R}$. We choose one of these two real embeddings of K as the designated "first" embedding $K \hookrightarrow \mathbb{R}$, whose image in $\mathbb{R}$ is denoted by $K^{(1)}$.

Stark's Conjecture for the extension K/F says that there exists an algebraic integer $\varepsilon \in \mathcal{O}_K$ that is an " L–function evaluator" just as before, namely:

$$L'(0,\chi_0) = 0 = -\frac{1}{2}\sum_{\rho\in G}\chi_0(\rho)\log|\rho(\varepsilon)^{(1)}| = -\frac{1}{2}\log|N_{K/F}(\varepsilon)^{(1)}|, \tag{3}$$

$$L'(0,\chi) = -\frac{1}{2}\sum_{\rho\in G}\chi(\rho)\log|\rho(\varepsilon)^{(1)}| = -\frac{1}{2}\log\left|\frac{\varepsilon^{(1)}}{\sigma(\varepsilon)^{(1)}}\right|. \tag{4}$$

In addition to ε being an L–function evaluator, Stark's Conjecture also says that $K(\varepsilon^{1/2})$ is an abelian extension of F, which holds iff $N_{K/F}(\varepsilon)$ is a square in K. By (3), $N_{K/F}(\varepsilon) = \pm 1$ and so the Stark unit ε is a "true" unit in this case. The abelian condition implies the stronger condition $N_{K/F}(\varepsilon) = 1$ since K has a real embedding.

Stark's Conjecture has been proved in general for relative quadratic extensions using methods that generalize the genus theory employed earlier. Since we can prove Stark's Conjecture for relative quadratic extensions, it is natural to try to extend this result to multiquadratic extensions.

If L/F is a relative Galois extension with Gal(L/F) isomorphic to a direct product of m copies of $\mathbb{Z}/2\mathbb{Z}$, we say that L/F is a multiquadratic extension of rank m. Sands has proved Stark's Conjecture when Gal(L/F) is the Klein 4-group, so the Conjecture is known when $m = 1, 2$.

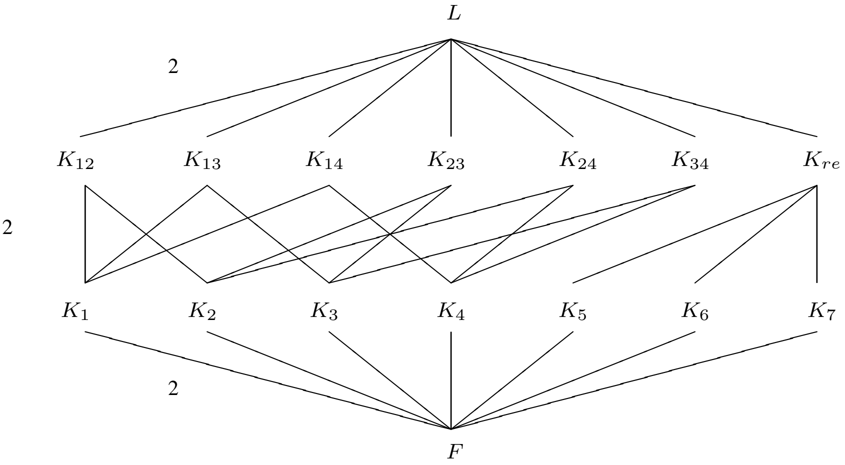
We proved in

[DST2] Dummit, D., Sands, J., Tangedal, B.:
Stark's conjecture in multi-quadratic extensions, revisited,
J. Théor. Nombres Bordeaux **15** (2003), 83–97,

that an algebraic integer $\varepsilon \in \mathcal{O}_L$ may always be found that is a simultaneous L–function evaluator for the L–functions associated to a multiquadratic extension L/F and in many cases we prove the abelian condition as well, namely, that $L(\sqrt{\varepsilon})$ is an abelian extension of F.

Our goal was to construct and study a special collection of multiquadratic extensions of rank 3 over real quadratic base fields for which the abelian condition is nontrivial and not known to hold by the theorems in [DST2].

If more than a certain number of finite primes of F ramify in L/F, the abelian condition is known to hold. Our examples were constructed in such a way that only a single prime $\mathfrak{p} \subset \mathcal{O}_F$ lying over 2 ramifies in L/F. The Hasse diagram for our examples looks as follows:



The 7 relative quadratic extensions over F are numbered in such a way that K_1, K_2, K_3, and K_4 all have signature $[2, 1]$ and K_5, K_6, and K_7 all have signature $[4, 0]$.

Corresponding to each K_j, $1 \le j \le 7$, is a generalized Kronecker symbol χ_j and a Stark unit $\varepsilon_j \in K_j$ ($\varepsilon_5 = \varepsilon_6 = \varepsilon_7 = 1$, but $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 are all nontrivial).

There are also precisely 7 quartic extensions of F contained in L all having Galois group isomorphic to the Klein 4-group over F. We label the first 6 of these as K_{ij}, with $1 \leq i < j \leq 4$, since each K_{ij} here is the composite of K_i and K_j. Each of these 6 fields has signature $[4, 2]$ and a nontrivial Stark unit $\varepsilon_{ij} \in K_{ij}$.

The 7th quartic extension of F contained in L is the composite of K_5, K_6, and K_7 and is denoted by K_{re} since it is totally real. The signature of L is $[8, 4]$.

By results obtained in [DST2], each ε_j is a square in K_j for $1 \leq j \leq 4$ and $\varepsilon_{ij} = \sqrt{\varepsilon_i}\sqrt{\varepsilon_j}$ for $1 \leq i < j \leq 4$.

Also, by [DST2], we have $\sqrt[4]{\varepsilon_j} \in L$ for $1 \leq j \leq 4$. Let $N_j = F(\sqrt[4]{\varepsilon_j})$ for $1 \leq j \leq 4$ and note that N_j is either a quartic or quadratic extension of F such that $F \subset K_j \subseteq N_j \subset L$. Since each N_j, $1 \leq j \leq 4$, can be one of exactly 4 distinct intermediate fields between F and L, there are 256 possible arrangements of the fields N_1, N_2, N_3, and N_4 inside L.

The following proposition holds independently of how the fields N_1, N_2, N_3, and N_4 are situated within L and represents the strongest result provable with respect to the examples we computed using the methods of [DST2].

Proposition

The element $\varepsilon = \sqrt[4]{\varepsilon_1}\sqrt[4]{\varepsilon_2}\sqrt[4]{\varepsilon_3}\sqrt[4]{\varepsilon_4} \in \mathcal{O}_L^\times$ *is a simultaneous* L*–function evaluator for all 8 of the* L*–functions* $L(s, \chi_j)$*,* $0 \le j \le 7$*, associated to the extension* L/F*.*

The reason that this proposition can be proved without any knowledge of how the fields N_j, $1 \leq j \leq 4$, are situated within L is that only the absolute values $|\rho(\varepsilon)^{(1)}|$, $\rho \in G :=$ Gal(L/F), appear in the L–function evaluation. However, the abelian condition requires for starters that $\rho(\varepsilon)^{(1)} = \prod_{j=1}^{4} \rho(\sqrt[4]{\varepsilon_j})^{(1)} > 0$ for all $\rho \in G$.

We will only have an *even* number of negative values among the numbers $\rho(\sqrt[4]{\varepsilon_j})^{(1)}$, $1 \leq j \leq 4$, for all $\rho \in G$ if the four fields N_1, N_2, N_3, and N_4 are situated within L in a *specific* fashion.

We know the “right” Stark unit for the extension L/F is $\varepsilon = \prod_{j=1}^{4} \sqrt[4]{\varepsilon_j}$, but what is the relation of these 4th roots of Stark units from quadratic subfields to each other?

The relationships among all of the various Stark units within L constitute what we refer to as the “functorial properties of Stark units”. Understanding these relationships is critical to proving the abelian condition in general and very few results have been obtained in this direction.

Our computations helped lead us to understand the exact requirements for the abelian condition to hold in this setting.

Theorem

If $\varepsilon = \prod_{j=1}^{4} \sqrt[4]{\varepsilon_j}$*, then* $L(\sqrt{\varepsilon})$ *is an abelian extension of* F *exactly when the fields* $N_j = F(\sqrt[4]{\varepsilon_j})$*,* $1 \le j \le 4$*, are arranged within* L *as follows:*

A.) All 4 fields N_j*,* $1 \le j \le 4$*, are quartic extensions of* F*.*
Assuming that $N_1 = K_{12}$ *(below,* 12 *is short for* K_{12}*, etc.),*
$(N_2, N_3, N_4) = (12, 34, 34), (23, 34, 14), (24, 13, 34)$*.*

B.) Exactly one of the N_j*'s is a quadratic extension of* F*.*
Assuming that $N_4 = K_4$*,*
$(N_1, N_2, N_3) = (12, 24, 23), (13, 23, 34), (14, 12, 13), (14, 24, 34)$*.*

C.) Exactly two of the N_j*'s are quadratic extensions of* F*.*
Assuming that $N_3 = K_3$ *and* $N_4 = K_4$*,*
$(N_1, N_2) = (12, 12)$*.*

D.) $N_j = K_j$ *for* $j = 1, 2, 3$ *and* 4*.*

The Stark unit ε is a square in L for class A examples when $(N_2, N_3, N_4) = (12, 34, 34)$, class B examples when $(N_1, N_2, N_3) = (14, 24, 34)$, and in case D. Otherwise, ε is *not* a square in L.

Comment: Having exactly 3 of the N_j's being quadratic extensions of F is incompatible with the abelian condition. Having all 4 of the N_j's being quadratic extensions of F is compatible with the abelian condition but no such example was found.

We computed 46 multiquadratic examples of rank 3 over real quadratic base fields and in all cases the abelian condition was verified to hold.

29 of our examples were of class A and in 9 of these examples, ε was a square in L.

10 of our examples were of class B and in 5 of these examples, ε was a square in L.

7 of our examples were of class C and ε was a nonsquare in L in all 7 cases.

In total, 32 of the 46 examples were such that ε was not a square in L and for these the abelian condition is not known to hold by any previous work.