Abelian varieties with prescribed embedding degree

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Overview

We construct *Weil numbers* that correspond to abelian varieties with prescribed *embedding degree*.

Overview:

- What is the embedding degree?
- What are Weil numbers and how to construct the corresponding abelian varieties?
- Our actual construction.

The embedding degree

- Let A be an abelian variety over a finite field $\mathbb{F} = \mathbb{F}_q$ and let $r \nmid q$ be a prime dividing $\#A(\mathbb{F})$.
- Two pairings:

$$\begin{array}{llcllcl} \text{Weil:} & A(\mathbb{F})[r] & \times & \widehat{A}(\mathbb{F})[r] & \rightarrow & \mu_r(\mathbb{F}), \\ \text{Tate:} & A(\mathbb{F})[r] & \times & \widehat{A}(\mathbb{F})/r\widehat{A}(\mathbb{F}) & \rightarrow & \mathbb{F}^*/(\mathbb{F}^*)^r \cong \mu_r(\mathbb{F}). \end{array}$$

- The embedding degree k of A with respect to r is the degree of the field extension $\mathbb{F}(\zeta_r)/\mathbb{F}$.
- For random r and q, the embedding degree grows like r.
- If k is small and the discrete logarithm problem is hard in both $A(\mathbb{F})[r]$ and $\mathbb{F}(\zeta_r)^*$, then these pairings can be used for pairing-based cryptography.

The embedding degree

The embedding degree of A with respect to $r \mid \#A(\mathbb{F})$ is the degree of $\mathbb{F}(\zeta_r)/\mathbb{F}$.

Lemma

The embedding degree of A with respect to r is equal to the order of $(q \bmod r)$ in $\mathbb{F}_r^*$.

Proof: The embedding degree is the smallest number k such that $r \mid \#\mathbb{F}_{q^k}^* = q^k - 1$. $\square$

So the embedding degree is k if and only if $(q \bmod r)$ is some primitive k-th root of unity in $\mathbb{F}_r$.

Weil numbers

- Let q be a prime power.
  A *Weil q-number* is an algebraic integer π such that $\pi\overline{\pi} = q$ for every embedding of π into $\mathbb{C}$.
- Honda-Tate theory gives a bijection

$$\frac{\{\text{simple abelian varieties over } \mathbb{F}_q\}}{\text{isogeny}} \leftrightarrow \frac{\{\text{Weil } q\text{-numbers}\}}{\text{conjugation}}$$
$$A \mapsto \mathrm{Frob}_q.$$

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If q is prime and $\pi \neq \pm\sqrt{q}$ is a Weil q-number, then

- $K = \mathbb{Q}(\pi)$ is a *CM field*, i.e. a non-real number field with a unique complex conjugation automorphism,
- the corresponding abelian variety A has dimension g, where $2g$ is the degree of K and
- $\#A(\mathbb{F}_q) = N_{K/\mathbb{Q}}(\pi - 1)$.

The CM method

Given a Weil q-number π, the corresponding abelian variety can be constructed using the *complex multiplication* method:

- List the isogeny classes of abelian varieties over $\overline{\mathbb{Q}}$ with CM by the ring of integers of $\mathbb{Q}(\pi)$.
- Reduce them modulo a prime dividing q.
- Some twist of one of the reduced varieties will have Frobenius π. Select the one of the correct order.

This method is only well-developed for dimensions 1 and 2 and some special cases of higher dimension and takes time exponential in the bit size of the discriminant of $\mathbb{Q}(\pi)$.

About our algorithm

We give an algorithm with

input:

- a positive integer k,
- a CM field K of degree $2g$ with a 'primitive CM type' and
- a prime $r \equiv 1 \pmod{k}$ that splits completely in K.

output:

a prime number q and a Weil q-number $\pi \in K$ corresponding to an abelian variety of dimension g with embedding degree k with respect to r.

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Heuristic expected run time polynomial in $\log r$ (for fixed K).

For $g = 1$, we recover the Cocks-Pinch algorithm, so we assume $g \geq 2$ for simplicity.

Special case: K cyclic

- Suppose ϕ generates $\mathrm{Gal}(K/\mathbb{Q})$ and $\mathfrak{r}$ is a prime of K dividing r. Let $\mathfrak{r}_i = \phi^{-i}(\mathfrak{r})$, so $r\mathcal{O}_K = \prod_{i=1}^{g} \mathfrak{r}_i\overline{\mathfrak{r}_i}$.
- We want $\pi \in \mathcal{O}_K$ with $q = \pi\overline{\pi} \in \mathbb{Z}$ prime such that
  1. $r \mid N_{K/\mathbb{Q}}(\pi - 1)$, e.g. $(\pi \bmod \mathfrak{r}) = 1 \in \mathbb{F}_r$ and
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- Idea: take $\pi = \prod_{i=1}^{g} \phi^i(\xi)$ with $\xi \in \mathcal{O}_K$, so $q = \pi\overline{\pi} = N_{K/\mathbb{Q}}(\xi) \in \mathbb{Z}$.

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- So all we need to do is find $\xi \in \mathcal{O}_K$ with prime norm and
  1. $\prod_{i=1}^{g}(\xi \bmod \mathfrak{r}_i) = 1$ and
  2. $\prod_{i=1}^{g}(\xi \bmod \overline{\mathfrak{r}_i}) = \zeta_k$ in $\mathbb{F}_r$.

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Algorithm

1. *Let* $\langle\phi\rangle = \mathrm{Gal}(K/\mathbb{Q})$, $\mathfrak{r} \mid r$ *a prime of* K *and* $\mathfrak{r}_i = \phi^{-i}(\mathfrak{r})$.
2. *Choose* α_i *and* β_i *randomly in* $\mathbb{F}_r^*$ *such that* $\prod \alpha_i = 1$ *and* $\prod \beta_i = \zeta_k$.
3. *Compute* $\xi \in \mathcal{O}_K$ *with* $(\xi \bmod \mathfrak{r}_i) = \alpha_i$ *and* $(\xi \bmod \overline{\mathfrak{r}_i}) = \beta_i$.
4. *If* $q = N_{K/\mathbb{Q}}(\xi)$ *is prime and* $\pi = \prod_{i=1}^{g} \phi^i(\xi)$ *generates* K, *return* π *and* q. *Otherwise, go to step (2).*

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The heuristic expected run time is polynomial in $\log r$ (fixed K).

Proof: As ξ is a lift of a random element modulo $r\mathcal{O}_K$, we expect its norm q to behave like r^{2g}. By the prime number theorem, we thus expect to need $\log(r^{2g})$ iterations before we find a prime q. Moreover, π generates K with probability tending to 1. $\square$

The type norm

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- A *CM type* of a CM field K of degree $2g$ is a set $\Phi = \{\phi_1, \ldots, \phi_g\}$ of embeddings of K into a normal closure L such that $\Phi \cup \overline{\Phi}$ is the complete set of embeddings.
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- The *type norm* N_Φ with respect to Φ is the map $\xi \mapsto \prod_{i=1}^{g} \phi_i(\xi)$.
  - Notice that for $\pi = N_\Phi(\xi)$, we have $\pi\overline{\pi} = N_{K/\mathbb{Q}}(\xi) \in \mathbb{Q}$.

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  - Notice that for $\pi = N_\Phi(\xi)$, we have $\pi\overline{\pi} = N_{K/\mathbb{Q}}(\xi) \in \mathbb{Q}$.
- The image of N_Φ does not lie in K but in a field called the *reflex field*.

The reflex field

- Given a pair (K, Φ) of a CM field and a CM type, there is a *reflex* pair $(\widehat{K}, \Psi)$.
  - The image of N_Φ lies inside $\widehat{K}$.
  - If Φ is primitive, then the reflex of $(\widehat{K}, \Psi)$ is (K, Φ).
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- We construct π as $N_\Psi(\xi)$ for some $\xi \in \mathcal{O}_{\widehat{K}}$.
- Remarks about the reflex field: (assume Φ is primitive)
  - If K is normal, then $\widehat{K} = K$.
  - In general, K and $\widehat{K}$ don't even have to have the same degree!
  - Denote the degree of $\widehat{K}$ by $2\widehat{g}$.
  - If $g = 2$, then $\widehat{g} = 2$. If $g = 3$, then $\widehat{g} \in \{3, 4\}$.

The general case

- Let $\Psi = \{\psi_1, \ldots, \psi_{\widehat{g}}\}$ be the reflex type.
- Let $\mathfrak{r}$ be a prime of $\mathcal{O}_L$ dividing r and $\mathfrak{r}_i = \psi_i^{-1}(\mathfrak{r}) \cap \mathcal{O}_{\widehat{K}}$. Then

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Algorithm

1. *Choose α_i and β_i randomly in $\mathbb{F}_r^*$ such that $\prod_{i=1}^{\widehat{g}} \alpha_i = 1$ and $\prod_{i=1}^{\widehat{g}} \beta_i = \zeta_k$ in $\mathbb{F}_r$.*
2. *Compute $\xi \in \mathcal{O}_{\widehat{K}}$ with $(\xi \bmod \mathfrak{r}_i) = \alpha_i$ and $(\xi \bmod \overline{\mathfrak{r}_i}) = \beta_i$.*
3. *If $q = N_{\widehat{K}/\mathbb{Q}}(\xi)$ is prime and $\pi = N_\Psi(\xi)$ generates K, return π and q. Otherwise, go to step (1).*

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- We expect our output to satisfy $\rho \sim 2g\widehat{g}$.
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- For fixed K, k and r, the optimal ξ gives $\rho \sim 2g$.
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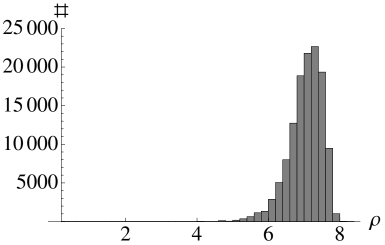
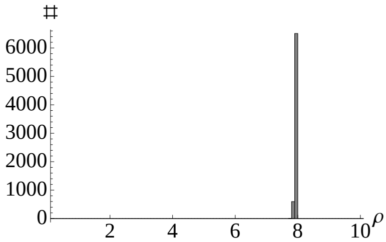
  - Open question: can we find it efficiently?
- A method by Freeman based on our algorithm, in which r is not prescribed, achieves $\rho < 2g\widehat{g}$ for some K and k.

Experimental results
$K = \mathbb{Q}(\zeta_5)$

Histograms of ρ-values produced by our algorithm:



$k = 2, r = 1021$
all possible α_1, β_1
minimal ρ: 4.19



$k = 10, r = 2^{160} + 685$
2^{20} random α_1 and β_1

Notice that $g = \widehat{g} = 2$.

Example
$K = \mathbb{Q}(\zeta_7)$, $k = 17$, $r = 2^{180} - 7427$

- Absolutely simple abelian varieties with CM by K are Jacobians of curves of the form $y^2 = x^7 + a$.
- Our algorithm found a suitable Weil q-number for

$$\begin{aligned} q \quad = \quad & 157558413811977153591787802014368793057776946867137463955067876140250081 2 \backslash \\ & 175974972634937716254216816917600718698808129260457040637146802812702044 0 \backslash \\ & 686127726925907718896620515610780682300009612087491561201718492420684320 4 \backslash \\ & 621759232946263357637192516979877402638911689714410855314811092763287402 9 \backslash \\ & 91115312604840826985712143103 3499 \quad \text{(1077 bits)} \end{aligned}$$

  in 51 seconds.
- It has $\rho = 17.95$ and $g = \widehat{g} = 3$.
- The corresponding curve is given by $y^2 = x^7 + 10$.

Summary

- Our algorithm constructs Weil numbers corresponding to abelian varieties over finite fields with prescribed embedding degree with respect to a subgroup of prescribed order r.
- We fix our CM field K in advance.
- The algorithm is polynomial in $\log r$.
- We get

$$\frac{\log \#A(\mathbb{F})}{\log r} \sim 2g\widehat{g}.$$