Construction of Special K3 Surfaces

Andreas-Stephan Elsenhans

Mathematisches Institut der Universität Göttingen

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joint work with Jörg Jahnel

K3 surfaces I

Definition (K3 surface)

A K3 surface is a simply connected proper algebraic surface with trivial canonical class.

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Examples

- A K3 surface of degree 2 is a twofold cover of $\mathbf{P}^2$, ramified at a smooth sextic.
- A K3 surface of degree 4 is a smooth quartic in $\mathbf{P}^3$.
- A K3 surface of degree 6 is a smooth complete intersection of a quadric and a cubic in $\mathbf{P}^4$.
- A K3 surface of degree 8 is a smooth complete intersection of three quadrics in $\mathbf{P}^5$.

K3 surfaces II

Properties of K3 surfaces

Betti numbers: 1, 0, 22, 0, 1

Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Picard group: $\mathbb{Z}^n$ *for* $n \in \{1, \ldots, 20\}$

Goal

To construct a K3 surface V of degree 2 over $\mathbb{Q}$ such that $\mathrm{Pic}(V) \cong \mathbb{Z}$.

That means:

- The equation of V has the form $w^2 = f_6(x, y, z)$.
  I.e., V is a double cover of $\mathbf{P}^2$ ramified at a smooth curve of degree 6.
- The geometric Picard group should be of rank 1.

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- The geometric Picard group should be of rank 1.

Additional condition:

- f_6 should be given in determinantal form.
  I.e., $f_6 = \det(M)$ for a symmetric matrix M.
  M should contain either linear or quadratic forms.

Upper bounds for the Picard rank

- Let V be a K3 surface over $\mathbb{Q}$ and p be a prime of good reduction. Then

$$\mathrm{Pic}(V_{\overline{\mathbb{Q}}}) \hookrightarrow \mathrm{Pic}(V_{\overline{\mathbb{F}}_p}) \hookrightarrow H^2_{\mathrm{et}}(V_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell).$$

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  A sufficiently large power of the Frobenius acts trivially on $\mathrm{Pic}(V_{\overline{\mathbb{F}}_p})$.
- The Frobenius actions on $\mathrm{Pic}(V_{\overline{\mathbb{F}}_p})$ and $H^2_{\mathrm{et}}(V_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$ are compatible.

$$\mathrm{rk}(\mathrm{Pic}(V_{\overline{\mathbb{Q}}})) \leq \mathrm{rk}(\mathrm{Pic}(V_{\overline{\mathbb{F}}_p})) \overset{(*)}{\leq} \# \left\{ \begin{array}{c} \text{Frobenius eigenvalues on} \\ H^2_{\mathrm{et}}(V_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \\ \text{which are of the form } p\zeta. \end{array} \right\}$$

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- Every prime of good reduction leads to an upper bound for $\mathrm{rk}(\mathrm{Pic}(V_{\overline{\mathbb{Q}}}))$. This bound is always even. The Tate conjecture implies equality in $(*)$.

The structure of the Picard group I

Let $V\colon w^2 = f_6(x, y, z)$ be a K3 surface.

- Divisors are constructed by pulling back a divisor of $\mathbf{P}^2$ and splitting.
- Assume that „$f_6(x, y, z) = 0$" allows a tritangent G. The pull-back of G splits into two projective lines D_1 and D_2.

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- The adjunction formula $2g - 2 = D(D + K) = D^2$ can be used to calculate the intersection product.

$$
\begin{aligned}
-2 &= D_1^2 = D_2^2 \\
2 &= 2G^2 = (D_1 + D_2)^2 = D_1^2 + D_2^2 + 2D_1D_2 \\
\Rightarrow\ 3 &= D_1D_2
\end{aligned}
$$

- Discriminant of the lattice $\langle D_1, D_2 \rangle$:

$$
\det \begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix} = 4 - 9 = -5\,.
$$

The structure of the Picard group II

Let $V\colon w^2 = f_6(x,y,z)$ be a K3 surface. Assume the existence of a conic Q which intersects „$f_6(x,y,z) = 0$" only with even multiplicities.

- The pull-back of Q splits into two projective lines C_1 and C_2.
- The pull-back of a line G leads to a divisor $D = \pi^*(G)$.
- Intersection products:

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\begin{aligned}
-2 &= C_1^2 = C_2^2 \\
2 &= 2G^2 = D^2 \\
4 &= 2QG = (C_1 + C_2)D \Rightarrow C_1 D = 2
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- Discriminant of the lattice $\langle C_1, D\rangle$:

$$\det \begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix} = -4 - 4 = -8\,.$$

- This lattice can not be refined to a lattice of discriminant -2.

How to prove $\mathrm{rk}(\mathrm{Pic}(V)) = 1$

Let $V\colon w^2 = f_6(x, y, z)$ be a K3 surface defined over $\mathbb{Q}$.

Assume

- $V_{\overline{\mathbb{F}}_3}$ has Picard rank 2 and „$f_6(x, y, z) = 0$“ has conic with only even intersection multiplicities over $\mathbb{F}_3$.
- $V_{\overline{\mathbb{F}}_5}$ has Picard rank 2 and „$f_6(x, y, z) = 0$“ has a tritangent over $\mathbb{F}_5$.

Then V is of geometric Picard rank 1.

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Proof:
The discriminants are not compatible.

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Proof:
The discriminants are not compatible.

See [Ronald van Luijk: Rational points on K3 surfaces, PhD Thesis (2005)].

Required algorithms

To construct a K3 surface of Picard rank 1, we need the following.

- An algorithm that searches for tritangents of „$f_6 = 0$“.
- An algorithm that searches for conics which intersect „$f_6 = 0$“ only with even multiplicities.
- An algorithm to compute the characteristic polynomial $\chi_\phi(t)$ of the Frobenius.

  More precisely, an algorithm to prove that $\chi_\phi(t)$ has at most two zeros of the form $p\zeta$ (counted with multiplicites).

  These two zeros correspond to the two divisors we know explicitly.

Searching for tritangents

How to search for tritangents of „$f_6(x, y, z) = 0$“:

- Consider a generic line $g\colon t \mapsto [1 : t : a + bt]$.
  $f_6(g(t))$ is a square in $\overline{K}[t]$ if and only if g is a tritangent of „$f_6 = 0$“.
- This leads to an algebraic system of equations.
  It can be solved by the computation of a Gröbner basis.
  The tritangents can be read off explicitly from the Gröbner basis.
- The remaining lines can be treated via the parametrizations $t \mapsto [1 : a : t]$ and $t \mapsto [0 : 1 : t]$.

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Remarks

- The computation time is less than 1 sec per example.
- From time to time, we find tritangents on randomly chosen examples.

Searching for conics

Searching for conics intersecting „$f_6(x, y, z) = 0$“ only with even multiplicities:

- Consider a conic $q\colon t \mapsto [q_1(t) : q_2(t) : q_3(t)]$.
  $f_6(q(t))$ is a square in $\overline{K}[t]$ if and only if all intersection multiplicities are even.
- This leads to an algebraic system of equations.
- The computation of a Gröbner basis fails.

Searching for conics defined over $\mathbb{F}_p$

Searching for conics defined over $\mathbb{F}_p$ intersecting „$f_6(x, y, z) = 0$“ only with even multiplicities:

- Build up a list of all smooth conics defined over $\mathbb{F}_p$.
- Factor $f_6(q(t))$ for each conic in the list.

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Remarks

- Instead of factoring $f_6(q(t))$, one could run an `IsSquare` routine.
- The number of all smooth conics is $p^5 - p^2$.
- The running time is $O(p^{5+\varepsilon})$.
- From time to time, we find such conics on randomly chosen examples.

The characteristic polynomial of the Frobenius

Let V be a K3 surface and ϕ be the Frobenius on $H^2_{\text{et}}(V_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$.

Theorem (Lefschetz trace formula)

$$Tr(\phi^d) = \#V(\mathbb{F}_{p^d}) - p^{2d} - 1$$

Let $\lambda_1, \dots, \lambda_{22}$ be the eigenvalues of ϕ on $H^2_{\text{et}}(V_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$.

$$Tr(\phi^d) = \lambda_1^d + \cdots + \lambda_{22}^d$$

Using Newton's identity, one can compute the symmetric functions σ_i of $\lambda_1, \dots, \lambda_{22}$. These are the coefficients of the characteristic polynomial χ_ϕ.

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Consequence: We have to count points on V.

How far does one have to count?

Let χ_ϕ be the characteristic polynomial of the Frobenius on $H^2_{\text{et}}(V_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$.

- Naively, one has to count the number of rational points over $\mathbb{F}_p, \ldots, \mathbb{F}_{p^{22}}$.

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- The functional equation $p^{22}\chi_\phi(t) = \pm\chi_\phi(\frac{p^2}{t})$ leads to a relation between σ_i and σ_{22-i}. Thus, it suffices to count the rational points over $\mathbb{F}_p, \ldots, \mathbb{F}_{p^{11}}$.

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- Using $\chi_\phi(p) = 0$, we get a linear relation for the coefficients of χ_ϕ. Hence, it suffices to count the rational points over $\mathbb{F}_p, \ldots, \mathbb{F}_{p^{10}}$.

How far does one have to count? II

We don't want to compute χ_ϕ. We want to prove that χ_ϕ has at most two zero of the form ζp.

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All roots of the characteristic polynomial χ_ϕ have absolute value p.

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Theorem (Deligne-Weil)

All roots of the characteristic polynomial χ_ϕ have absolute value p.

This theorem can be used to disprove that a given polynomial is the characteristic polynomial of the Frobenius for any K3 surface. Just calculate the roots as floating point numbers.

How to count points fast?

Let $V\colon w^2 = f_6(x,y,z)$ be a K3 surface defined over $\mathbb{F}_p$.

- $\#V(\mathbb{F}_q) = \sum_{[x:y:z]\in\mathbf{P}^2(\mathbb{F}_q)} 1 + \chi_2(f_6(x,y,z))$

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  This requires q^2+q+1 evaluations of f_6 and χ_2.

- V is defined over $\mathbb{F}_p$. Therefore, the summands for $[x : y : z]$ and $[\phi(x) : \phi(y) : \phi(z)]$ coincide.

  It is enough to inspect a fundamental domain of the Frobenius.

  We gain a factor of $[\mathbb{F}_q : \mathbb{F}_p]$.

Counting points in a decoupled situation

Let $V\colon w^2 = f_6(x,y,z)$ be a K3 surface defined over $\mathbb{F}_p$.
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- The double sum has approximately $q^2(1-\frac{1}{e})^2$ summands such that $\#g^{-1}(u) > 0$ and $\#h^{-1}(v) > 0$.

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- The double sum has approximately $q^2(1 - \frac{1}{e})^2$ summands such that $\#g^{-1}(u) > 0$ and $\#h^{-1}(v) > 0$.
- For evaluation, one can build up tables stating how often g and h represent which value. Then, the evaluation reduces to one addition in $\mathbb{F}_q$ per summand.

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- The double sum has approximately $q^2(1-\frac{1}{e})^2$ summands such that $\#g^{-1}(u) > 0$ and $\#h^{-1}(v) > 0$.
- For evaluation, one can build up tables stating how often g and h represent which value. Then, the evaluation reduces to one addition in $\mathbb{F}_q$ per summand.
- Additionally, one table can be restricted to a fundamental domain of the Frobenius.
- The resulting algorithm is approximately 10 times faster than that for the general case.

A determinantal example with linear forms I

Example

$V: w^2 = \det(xM_x + yM_y + zM_z)$ *is a K3 surface of geometric Picard rank* 1.

$$
\begin{aligned}
f_6(x,y,z) = \det(xM_x + yM_y + zM_z) = \\
& - 106613x^6 - 8455829x^5y + 3479390x^5z - 5386183x^4y^2 \\
& - 10075660x^4yz + 3946878x^4z^2 + 3130266x^3y^3 - 10182480x^3y^2z \\
& - 858915x^3yz^2 + 582012x^3z^3 + 2787726x^2y^4 + 1029050x^2y^3z \\
& - 776790x^2y^2z^2 + 3132270x^2yz^3 - 1365782x^2z^4 - 4062988xy^5 \\
& - 4529440xy^4z - 4108065xy^3z^2 + 1753440xy^2z^3 + 2845590xyz^4 \\
& - 388050xz^5 - 448780y^6 + 3612620y^5z + 4818675y^4z^2 \\
& - 1646250y^3z^3 - 2435045y^2z^4 + 275960yz^5 + 30699z^6
\end{aligned}
$$

A determinantal example with linear forms II

The matrices:

$$M_x := \begin{pmatrix} 3 & 9 & 3 & 9 & 14 & 11 \\ 9 & 13 & 0 & 7 & 6 & 10 \\ 3 & 0 & 14 & 10 & 13 & 5 \\ 9 & 7 & 10 & 7 & 1 & 13 \\ 14 & 6 & 13 & 1 & 5 & 2 \\ 11 & 10 & 5 & 13 & 2 & 6 \end{pmatrix}$$

$$M_y := \begin{pmatrix} 0 & 5 & 13 & 13 & 5 & 1 \\ 5 & 2 & 5 & 11 & 8 & 11 \\ 13 & 5 & 0 & 10 & 13 & 0 \\ 13 & 11 & 10 & 13 & 10 & 10 \\ 5 & 8 & 13 & 10 & 5 & 11 \\ 1 & 11 & 0 & 10 & 11 & 3 \end{pmatrix} \quad M_z := \begin{pmatrix} 2 & 1 & 1 & 1 & 5 & 0 \\ 1 & 10 & 9 & 4 & 14 & 4 \\ 1 & 9 & 2 & 13 & 10 & 2 \\ 1 & 4 & 13 & 6 & 12 & 7 \\ 5 & 14 & 10 & 12 & 8 & 11 \\ 0 & 4 & 2 & 7 & 11 & 0 \end{pmatrix}$$

A determinantal example with linear forms III

Reduction modulo 3

- *Conic:* $xz + y^2 + 2yz + 2z^2 = 0$,
  Parametrization $q\colon u \mapsto [2u^2 + u + 1 : u : 1]$,
  $f_6(q(u)) = (u^2 + u + 2)^2(u^4 + u + 2)^2$.
- *Traces on* $H^2_{\mathrm{et}}(V_{\overline{\mathbb{F}}_3}, \overline{\mathbb{Q}}_\ell)$*:*
  3, 9, 54, −27, 648, 3 888, 10 692, 32 805, 100 602, 463 644.
- *Scaled characteristic polynomial of the Frobenius:*

$$\begin{aligned}(t-1)^2(3t^{20} + 3t^{19} + 3t^{18} + 2t^{17} + 3t^{16} + 2t^{15} \\ - 2t^{13} - 3t^{12} - 4t^{11} - 6t^{10} - 4t^9 - 3t^8 \\ - 2t^7 + 2t^5 + 3t^4 + 2t^3 + 3t^2 + 3t + 3)/3\,.\end{aligned}$$

A determinantal example with linear forms IV

Reduction modulo 5

- *Tritangent:* $x = 0$
- *Traces on* $H^2_{\mathrm{et}}(V_{\overline{\mathbb{F}}_5}, \overline{\mathbb{Q}}_\ell)$*:*

  11, 41, 86, 1 081, −2 024, 11 396, 418 716, 1 298 561, 8 089 556, −19 047 174.
- *Scaled characteristic polynomial of the Frobenius:*

$$(t-1)^2(5t^{20} - t^{19} + t^{18} + 2t^{17} + 3t^{15} + t^{14} - 2t^{13} + t^{12} - t^{11} + 2t^{10} - t^9 + t^8 - 2t^7 + t^6 + 3t^5 + 2t^3 + t^2 - t + 5)/5\,.$$

An determinantal example with quadratic forms I

Example

$V: w^2 = \det(M(x,y,z))$ *is a K3 surface of geometric Picard rank* 1.

$$
\begin{aligned}
m_{11} &= 9x^2 + 5xy + 9xz + 11y^2 + 10yz \\
m_{12} = m_{21} &= 2x^2 + 10xy + 10xz + 10yz + 8z^2 \\
m_{13} = m_{31} &= 14x^2 + 12xy + 7xz + 9y^2 + 13yz + 7z^2 \\
m_{22} &= x^2 + 4xy + 9y^2 + 6yz + 3z^2 \\
m_{23} = m_{32} &= 4xy + 12xz + y^2 + 4yz + 14z^2 \\
m_{33} &= 14x^2 + 5xy + 6xz + 5y^2 + 5yz + 3z^2
\end{aligned}
$$

An determinantal example with quadratic forms II

$$
\begin{aligned}
f_6(x,y,z) = \det(M) = \\
&- 196x^6 - 1078x^5y + 56x^5z - 2254x^4y^2 - 320x^4yz \\
&+ 546x^4z^2 - 2369x^3y^3 - 1410x^3y^2z + 3145x^3yz^2 \\
&+ 1354x^3z^3 - 1873x^2y^4 - 2390x^2y^3z + 2180x^2y^2z^2 \\
&+ 6360x^2yz^3 + 882x^2z^4 - 1241xy^5 - 3225xy^4z \\
&- 1530xy^3z^2 + 4225xy^2z^3 + 6060xyz^4 + 2031xz^5 \\
&- 245y^6 - 1235y^5z - 2700y^4z^2 - 1515y^3z^3 + 475y^2z^4 \\
&+ 1810yz^5 + 1229z^6
\end{aligned}
$$

An determinantal example with quadratic forms III

Reduction modulo 3

- *Conic:* $x^2 + xy + 2xz + z^2 = 0$,

  Parametrization: $q\colon u \mapsto [u^2 : 2 : 2u^2 + 2u]$,

  $f_6(q(u)) = (u+1)^2(u^5 + u^4 + u^3 + u + 1)^2$.
- *Traces on* $H^2_{\mathrm{et}}(V_{\overline{\mathbb{F}}_3}, \overline{\mathbb{Q}}_\ell)$*:*
  5, 7, 71, 103, 65, −305, −625, −16 769, 129 734, −28 823.
- *Scaled characteristic polynomial of the Frobenius:*

$$(t-1)^2(3t^{20} + t^{19} + 2t^{18} + t^{16} + t^{15} + 2t^{14} + 2t^{13} + 3t^{12} + 2t^{10} + 3t^8 + 2t^7 + 2t^6 + t^5 + t^4 + 2t^2 + t + 3)/3\,.$$

An determinantal example with quadratic forms IV

Reduction modulo 5

- *Tritangent:* $x = 0$
- *Traces on* $H^2_{\mathrm{et}}(V_{\overline{\mathbb{F}}_5}, \overline{\mathbb{Q}}_\ell)$*:*
  8, 44, −103, 1 236, 3 043, −7 891, −495 683, 970 196, 12 359 273, −11 502 651.
- *Scaled characteristic polynomial of the Frobenius:*

$$(t-1)^2(5t^{20} + 2t^{19} + t^{18} + 5t^{17} + 2t^{16} + 2t^{15} + 5t^{14} + 8t^{13} + 4t^{12} + 2t^{11} + 8t^{10} + 2t^9 + 4t^8 + 8t^7 + 5t^6 + 2t^5 + 2t^4 + 5t^3 + t^2 + 2t + 5)/5\,.$$

Details of the construction I

We construct $\mathbb{F}_3$ examples and $\mathbb{F}_5$ examples separately and combine them via the Chinese remainder theorem.

Algorithm for the $\mathbb{F}_3$ part:

- Choose M randomly symmetric and calculate $f_6 = \det(M)$.
- Test the smoothness of „$f_6(x, y, z) = 0$“ using a Gröbner basis.
- Search for tritangents and conics with even intersection multiplicities.
- If there is no tritangent and exactly one such conic then count the rational points of the K3 surface defined over $\mathbb{F}_3, \ldots, \mathbb{F}_{3^{10}}$.
- Calculate and factorize the characteristic polynomial of the Frobenius.
- We succeed if the degree 20 factor contains no cyclotomic polynomial.

Details of the construction II

The $\mathbb{F}_5$ examples can only be handled in the decoupled case. Chose some of the entries of M randomly such that decoupling leads to a linear system for the remaining coefficients.

- Test smoothness of „$f_6(x, y, z) = 0$“ using a Gröbner basis.
- Search for tritangents and conics with even intersection multiplicities.
- If there is exactly one tritangent and no such conic then count the rational points of the K3 surface defined over $\mathbb{F}_5, \dots, \mathbb{F}_{5^9}$.
- Try to prove $\text{rk}(\text{Pic}(V_{\overline{\mathbb{F}}_p})) = 2$.
- If the proof is successful then we count the rational points defined over $\mathbb{F}_{5^{10}}$.
- Calculate and factorize the characteristic polynomial of the Frobenius.

Summary

Conclusion

- Found explicit examples of K3 surfaces over $\mathbb{Q}$ of degree 2 and geometric Picard rank 1.
- Calculations are based on reduction modulo p methods.
- Methods are flexible enough to construct many examples.
  We illustrated this by constructing examples in determinantal form.

Estimates for the Picard rank in characteristic p

- Lower bounds by explicitly constructing divisors.
- Upper bounds by point counting.
- Provable ranks are always even.

Bounds for characteristic zero

- Reduction modulo p induces an injection on the Picard lattices.
- Incompatibility of the discriminants reduces the rank.