MINIMAL HEIGHTS AND REGULATORS FOR ELLIPTIC SURFACES

ANTS VIII Poster Session Abstract

Sonal Jain

New York University Courant Institute of Mathematical Sciences
251 Mercer Street, New York NY 10012
jain@cims.nyu.edu

Let K be a number field or function field. Let E be an elliptic curve over K, nonconstant in the case K is a function field. Néron and Tate independently showed that there is a canonical height function $\hat{h} : E(K) \longrightarrow [0, \infty)$. In the case that K is a function field, $\hat{h}$ takes its values in $\mathbb{Q}$. The quotient $E(K)/E(K)_{tors}$ is a finitely generated free abelian group on which $\hat{h}$ descends to a positive definite quadratic form. If P is a non-torsion point, one may ask: How small can $\hat{h}(P)$ be? A conjecture by Lang postulates a uniform lower bound for the canonical height of non-torsion points on elliptic curves.

Conjecture. *(Lang) If $K = \mathbb{Q}$, then $\hat{h}(P) \gg \log |\Delta_E|$, and more generally if K is number field, then $\hat{h}(P) \geq C_K \log |N_{K/\mathbb{Q}}\Delta_{E/K}|$. Over $\mathbb{C}(t)$ or $\mathbb{C}(C)$ for C a curve, the same bound holds with $\log |N_{K/\mathbb{Q}}\Delta_{E/K}|$ replaced by the discriminant degree $12n$.*

This conjecture was proven by Hindry-Silverman in 1988 under the hypothesis of the ABC conjecture of Masser-Oesterlé [3]. Mason had already proved the ABC conjecture over function fields [4], and hence Lang's conjecture is true for function fields. Thus, it is natural to ask: What is the length of the shortest vector in the lattice $E(K)/E(K)_{tors}$, i.e. what is the best possible value of C_K?

Hindry and Silverman determined an explicit value for C_K that was approximately $6 \cdot 10^{-11}$. In 2002 Elkies, using two new ideas, improved the value of C_K to about $1/25330 \approx 3.9 \cdot 10^{-5}$ and conjectured what the best value of C_K should be [2]. First, he proved that for purposes of minimizing the canonical height $\hat{h}$, it suffices to consider only curves with semistable reduction. Second, he showed Hindry-Silverman's estimation of $\hat{h}(P)$ can be vastly improved by viewing the problem as one of linear programming.

In this poster we show how to generalize Elkies' methods to elliptic curves of rank 2. If E/K has rank bigger than 1, The determination of the constant C_K appearing in Lang's Conjecture gives one a lower bound for the volume of the fundamental domain of the lattice $E(K)/E(K)_{tors}$. It is natural to ask: what smallest possible volume of a fundamental domain for the lattice $E(K)/E(K)_{tors}$?

Given a basis reduced (P, Q) of the lattice $E(K)/E(K)_{tors}$, we show how one can use linear programming to compute a lower bound for any positive form $a\hat{h}(P) + b\langle P, Q\rangle + c\hat{h}(Q)$, i.e. a, b, c satisfy $0 < c, -c < a, -(a + c)/2 < b$. For example, we prove the following:

Theorem. *Let (P, Q) be a reduced basis for a rank 2 subgroup of an elliptic surface of discriminant degree $12n$ over $\mathbb{P}^1(\mathbb{C})$, so $\hat{h}(P) \leq \hat{h}(Q)$. The canonical height of Q is at least $12nC_0$, where $C_0 = 1/3595$. If one replace $\mathbb{P}^1(\mathbb{C})$ by a curve of C of genus g, then $\hat{h}(Q) \geq (12n)C_0 - (g - 1)D_0$ for some absolute constant D_0.*

Using the lower bound for $\hat{h}(P)$, the above result immediately gives a new lower bound for the regulator $R(P,Q) \geq \hat{h}(P)\hat{h}(Q) - \hat{h}(P)^2/4$.

Furthermore, we normalize by the discriminant degree $12n$ to find lower bounds $B_{(a,b,c)}$ for thousands of forms $a\hat{h}(P) + b\langle P,Q\rangle + c\hat{h}(Q)$. Each lower bound determines a plane $ax + by + cz \geq B_{(a,b,c)}$ in the 3-dimensional space of reduced 2-dimensional quadratic forms. The asymptotically obtainable region sits on one side of the form, and by minimizing enough forms one can restrict the shape of the region in $\mathbb{R}^3$.

Using parameter counting heuristics, we are able to improve each lower bound and compute what should be supporting planes for the region. For example, the best possible value for the constant C_0 in the theorem above is, conjecturally, 19/5059. We use the data generated by computing thousands of supporting planes to draw the actual shape of the asymptotically obtainable region in $\mathbb{R}^3$. Furthermore, because the linear program changes in a predictable fashion as we vary the parameters a, b and c, we are able to deduce that the boundary of the region must be piecewise smooth. We make a conjecture about the actual shape of the region, and attempt to identify the elliptic surface with the minimal regulator as a point on the boundary of the region.

References

[1] Cox, D.A., Zucker, S.: Intersection Numbers of Elliptic Surfaces. *Inventiones Mathematicae* **53** (1979), 1-44.

[2] Elkies, N.D.: Points of Low Height on Elliptic Curves and Surfaces I: Elliptic surfaces over $\mathbb{P}^1$ with small d.

[3] Hindry, M., Silverman, J.H.: The canonical height and integral points on elliptic curves. *Inventiones Mathematicae* **93** (1988), 419-450.

[4] Mason, R.C.: *Diophantine Equations over Function Fields*, London Math. Soc. Lect. Note Ser. **96**, Cambridge Univ. Press 1984.

[5] PARI/GP, versions 2.1.14. Bordeaux, 2000-4, `http://pari.math.u-bordeaux.fr`.

[6] Shioda, T.: Some remarks on elliptic curves over function fields. *Astérisque* **209** (1992) *proceedings of Journees Arithmetiques* 1991 (Genéve), D.F. Coray and Y.-F. S. Petermann, eds., 99-114.

[7] Silverman, J.H.: Computing Heights on Elliptic Curves. *Math. of Computation* **51** #183 (July 1988), 339-358.

[8] Szpiro, L.: Discriminant et conducteur des courbes elliptiques. *Astérisque* **183** (1990) *Seminaire sur les Pinceaux de Courbes Elliptiques, Paris* 1988, 7-18.