Minimal Heights and Regulators for Elliptic Surfaces

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1. Lang's Conjecture

LET E/K be an elliptic curve, K a number field or complex function field

- Néron and Tate showed (independently) there is a **canonical height function**:

$$\hat{h} : E(K) \longrightarrow [0, \infty)$$

- If K is a function field, $\hat{h}(P) \in \mathbb{Q}$
- $E(K)/E(K)_{tors} \cong \mathbb{Z}^r$ finitely generated free abelian group
  $\implies \hat{h}$ descends to positive definite quadratic form on $E(K)/E(K)_{tors}$

QUESTION: If P is nontorsion, how small can $\hat{h}(P)$ be?

Conjecture (Lang) If $K = \mathbb{Q}$, then $\hat{h}(P) \gg \log |\Delta_E|$. If K is number field, then

$$\hat{h}(P) \geq C_K \log |N_{K/\mathbb{Q}} \Delta_{E/K}|$$

Over $\mathbb{C}(t)$ or $\mathbb{C}(C)$ for C a curve, the same bound holds with $\log |N_{K/\mathbb{Q}} \Delta_{E/K}|$ replaced by the discriminant degree $12n$

2. A Theorem of Hindry-Silverman

HINDRY-SILVERMAN proved (1988):

ABC conjecture $\implies$ Lang's conjecture

For K function field, Hindry-Silverman determine explicit value for $C_K \cong 6 \cdot 10^{-11}$

Basic Idea in Hindry-Silverman proof:

- Replace P by $12P$ so P meets every additive fiber at identity component (incurs factor of 12^2 in lower bound)
- Canonical height $\hat{h}(P)$ depends on heights of $\hat{h}(mP)$ for integers m
- Using facts about elliptic surfaces, choose coefficients c_m so that

$$\sum_{m=1}^{\infty} c_m \hat{h}(mP) \geq 12n \qquad (1)$$

- Obtain a lower bound

$$\hat{h}(P) \geq 12n / (\sum_{m=1}^{\infty} m^2 c_m)$$

3. Elkies' Approach

ELKIES (2002) used two new ideas to improve the lower bound:

- **Semistable reduction:** When minimizing $\hat{h}(P)$, may assume semistable reduction
- Instead of choosing c_m, use theory of heights on elliptic surfaces to find linear constraints on the c_m s.t. (1) holds for c_m in some feasible region
- **Linear programming:** Minimize $\sum_{m=1}^{\infty} m^2 c_m$ (increases bound $\approx$ 5000x)

- This improves the constant to $C_K \approx 1/25330$, or $\hat{h}(P) \geq n/2111$
- Using heuristic arguments to reduce size of the feasible region, Elkies conjectures best possible $C_K \approx 1/3520$

4. Generalization to Rank 2

WE GENERALIZE Elkies' approach to the case in which E has rank at least 2:

QUESTION: What is smallest possible regulator for a rank 2 subgroup?
What are the possible lattices generated by 2 independent sections?

- **Semistable reduction**: Need only consider curves with semistable reduction
- Next, fix a, b and find lower bound for $a\hat{h}(P) + b\langle P, Q\rangle + \hat{h}(Q)$:

$$\sum_{m,n} c_{m,n} \hat{h}(mP + nQ) = \sum_{m,n} m^2 c_{m,n} \hat{h}(P) + \sum_{m,n} 2mnc_{m,n} \langle P, Q\rangle + \sum_{m,n} n^2 c_{m,n} \hat{h}(Q)$$

- Add the conditions corresponding to P, Q forming a reduced basis, and write

$$(\sum_{m,n} m^2 c_{m,n} - \alpha + \gamma)\hat{h}(P) + (\sum_{m,n} 2mnc_{m,n} - 2\gamma + \beta)\langle P, Q\rangle + (\sum_{m,n} n^2 c_{m,n} + \alpha)\hat{h}(Q) \qquad (2)$$

  where $\alpha, \beta, \gamma \geq 0$
- Use theory of heights on elliptic surfaces to constrain the $c_{m,n}$ s.t.

$$\sum_{m,n} c_{m,n} \hat{h}(mP + nQ) \geq 12n$$

- Add two linear conditions to ensure ratio of coefficients in (2) are a and b
- **Linear Programming:** Minimizing $(\sum_{m,n} n^2 c_{m,n} + \alpha)$ yields a lower bound of

$$a\hat{h}(P) + b\langle P, Q\rangle + \hat{h}(Q) \geq 12n / (\sum_{m,n} n^2 c_{m,n} + \alpha) \qquad (3)$$

5. New Results

- If Q is the point of second smallest height, constant in Lang's Conjecture improves to $1/3595$ (conjecturally $19/5059$), i.e.

$$\hat{h}(Q) \geq n/300$$

- Lower bound for the form $\hat{h}(Q) - \hat{h}(P)/4$ is $n/400$
- Since $\{P, Q\}$ reduced, volume of fundamental domain for lattice is bounded below by

$$V = \hat{h}(P)\hat{h}(Q) - \langle P, Q\rangle^2 \geq \hat{h}(P)\hat{h}(Q) - \hat{h}(P)^2/4$$

- New lower bound:

$$V = (\det \hat{h})(P, Q) \geq n^2 / (2111 \cdot 400)$$

- Minimize (3) to obtain a plane in $aX + bY + Z \geq C$ in 3-dimensional space of reduced 2-dimensional quadratic forms
- Find enough planes to restrict shape of the region
- Use heuristics to reduce size of feasible region of each linear program
- Find **supporting plane** for the region in each direction

6. The Region of Obtainable Rank 2 Forms

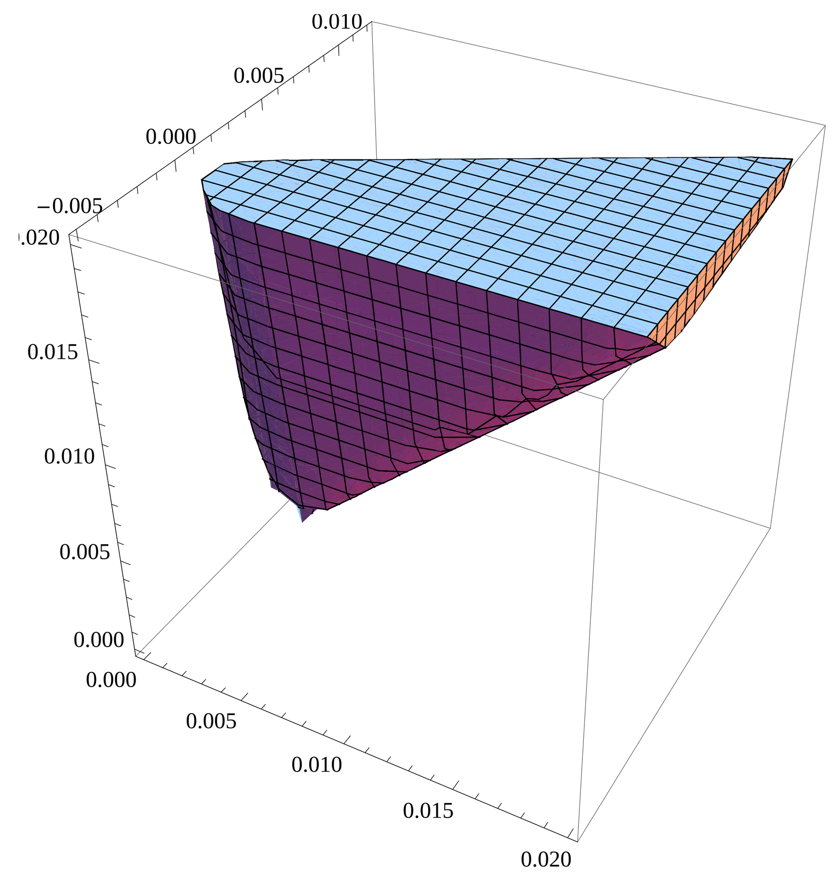

Figure 1: *As we increase the number of planes, we approach the asymptotically obtainable region in $\mathbb{R}^3$. The boundary must be piecewise algebraic.*

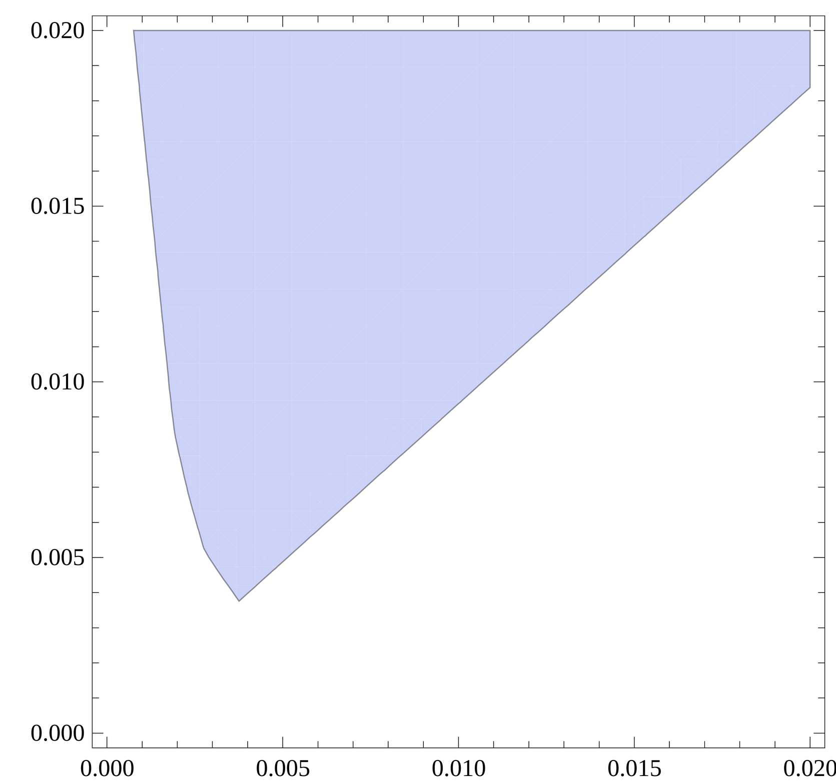

Figure 2: *We draw the intersection of the region with the plane $\langle P, Q\rangle = 0$.*

References

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