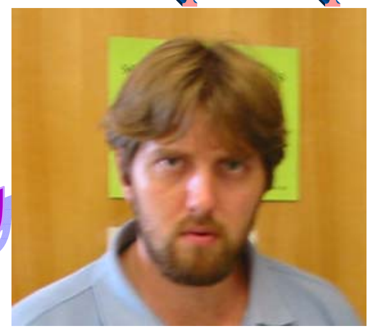
Running time predictions for square products,

Ernie Croot,



Robin Pemantle,

Prasad

Tetali,

& Andrew Granville,

and large prime variations

Running time predictions for square products, and large prime variations

by

Ernie Croot,

Andrew Granville,

Robin Pemantle,

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Factoring algorithms

- Dixon's random squares algorithm
- The quadratic sieve
- Multiple polynomial quadratic sieve
- The number field sieve

. All have a common central idea

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Idea: Generate a pseudo-random sequence of integers a_1, a_2, ..., with each

$$a_i \equiv b_i^2 \pmod{n},$$

until the product of a subseq of the a_i's is a square,

$$\text{say } Y^2 = a_{i_1} \cdots a_{i_k}.$$

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$$n \mid Y^2 - X^2 = (Y - X)(Y + X).$$

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$\geq 50\%$ CHANCE $\gcd(n, Y - X)$ IS A NON-TRIVIAL FACTOR OF n.

Analysis of running times

1994 ICM, Pomerance: In the (heuristic) *running time analysis* of such factoring algorithms one assumes that the pseudo-random sequence $a_1, a_2, \ldots$ is close enough to random, to make predictions based on this assumption.

. Why not study this as an abstract problem?

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Pomerance's problem

Select integers $a_1, a_2, \cdots \leq x$ independently at random; i.e.

$$\text{Prob}(a_j = m) = \frac{1}{x} \quad \forall\, 1 \leq m \leq x;$$

Stop at a_T as soon as there exists

$$a_{i_1} \cdots a_{i_k} = Y^2$$

What is expected stopping time?

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Interesting because...

- If this expected stopping time is far less than what is obtained by the algorithms currently used, *maybe* we can speeding up factoring algorithms.

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- Possibility of proving something without assumption.

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n is y-*smooth* if $p|n \Rightarrow p \leq y$
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Schroeppel-Pomerance: With probability going to 1, we have

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Recently, in probability theory, results showing "*sharp transitions*"; i.e. random algorithms tend to stop at a certain precise time with probability going to 1.

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Conjecture: $f(x) = e^{-\gamma}J_0(x)$, i.e.

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Theorem: Almost proved it!

$$\left(\frac{\pi}{4} - \epsilon\right) e^{-\gamma}J_0(x) < T \; < \; (1+\epsilon)e^{-\gamma}J_0(x),$$

with probability going to 1.

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Theorems: With prob going to 1:

- $\frac{\pi}{4}\left(e^{-\gamma} - \epsilon\right) J_0(x) < T \; < \; (e^{-\gamma}+\epsilon)J_0(x).$

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- If $T < \left(\frac{\pi}{4}-\epsilon\right) e^{-\gamma} J_0(x)$ then the square product is just one a_i, a square.

Schroeppel's 1985 approach

Study only the y-smooth a_i's.

Factor each as $a_i = 2^{e_{i,1}} 3^{e_{i,2}} \ldots p_k^{e_{i,k}}$

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Technique works, with prob 1, for

$$T > (1+\epsilon)J_0(x)$$

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Factor base: Primes up to y

Keep only y-smooth a_i's

Square product after $\geq k$ such a_i's

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New result: By time $(1+\epsilon)J_0(x)$ one has not one square product, but *many*, about $\epsilon\pi(y_0)$, with prob $\to 1$.

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Key Variant: Large Prime Variation

Also keep $a_i = pb_i$ where b_i is y-smooth and p is a prime $> y$.

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Theorem: Speed up by factor

$$.74997591747934498263\ldots \approx \frac{3}{4}$$

. Will prove this later

Multiple large prime variations

Can try two large primes: $a_i = pqb_i$,
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PROVED SPEED-UPS

ℓ	$M = \infty$	$M = 100$	$M = 10$
0	1	1	1
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2	.6415	.6448	.6745
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For ℓ-large primes variation, each large prime in (y, My).

. Speed-up for # of a_i's searched, not for factoring algorithm

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How big are y_0 and J_0?

For $L(x) = e^{\sqrt{\frac{1}{2}\log x \log\log x}}$,

$y_0(x) = L(x)^{1+o(1)}$ and $J_0(x) = L(x)^{2+o(1)}$

. These estimates can be made more precise but not to asymptotics

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But still cannot estimate J_0 accurately – maybe value can be computed (Bernstein) in any example.

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$\therefore$ Expected # of k-tuples, from the $T = \eta J_0$ values of a_i, that are p times a y-smooth:

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Analysis; one large prime, I

$$\text{Prob}(a = pb \le x: \; b \text{ is } y-\text{smooth}) = \frac{\Psi(x/p,y)}{x}$$

$$= \frac{\Psi(x/p,y)}{\Psi(x,y)} \cdot \frac{\Psi(x,y)}{x} \sim \frac{\log y}{p} \cdot \frac{\Psi(x,y)}{x}$$

for $y < p < y \log y$.

$\therefore$ Expected # of k-tuples, from the $T = \eta J_0$ values of a_i, that are p times a y-smooth:

$$\sim \binom{T}{k} \left(\frac{\log y}{p} \cdot \frac{\Psi(x,y)}{x} \right)^k \sim \frac{1}{k!} \left(\frac{\eta y}{p} \right)^k$$

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Constant = 1 for $\eta = .74997591747934498263\ldots$

 And for two large primes?

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How do we count the different type of matrices that arise?

A graph theory approach

Construct graph $G = G(M)$:
For each a_i a vertex $v_i \in G$,
With $v_i \sim v_I$ of colour p_j if $p_j | (n_i, n_I)$
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— Inspired combinatorics of 'species',

— Central to gas thermodynamics

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Sadly we re-organized a non-abs cvgent series in our proof, and we have been unable to fix this proof!

In fact we swapped order of summation in applying

$$\#\text{rows} - \text{rank}(M_R) = \sum_{\substack{S \subset R \\ M_S \in \mathcal{M}}} (-1)^{\#\text{ones}(S)}$$

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Hypergraphs to the rescue

Vertices correspond to primes in (y, My), and *hyperedges* to the odd power prime factors of each a_i.

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And the same divergence now implies one gets many square products soon after getting the first!

Pomerance's application

If one optimizes parameters in Pomerance's problem, then the running time of the factoring algorithm is dominated by finding the square product (i.e. *the matrix step*).

Matrix step: Wiedemann or Lanczos take time

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much smaller than y_0, so Pomerance's problem not so useful as had appeared! Expected running time is:

$$J_1 := J_0 \; y_0^{(1+o(1))/(\log\log x)^2}$$

Practical speed-ups for random squares algorithm from large prime variations

Reduction in T obtained:

ℓ	$M = \infty$	$M = 100$	$M = 10$
0	1	1	1
1	.7499	.7517	.7677
2	.6415	.6448	.6745
3	.5962	.6011	.6422
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If reduction in T here is a factor η, then the random squares algorithm is sped up by a factor

$$\approx \frac{1}{(\log x)^{\log(1/\eta)}}.$$

What practical people say

Practical considerations (not theory):

Design of the computer,

Language & implementation,

Memory issues (large arrays), swaps

How "reports" handled

Programmer's prejudices and prior experiences.

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If M is large, then more a_j's, more pseudosmooths, but slows down sieving (as there are more primes to test).

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$.4$ to $.5$ for $k = 1$;
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For $x \approx 10^{180}$ our predictions: Speed-ups of $.41, 0.25, 0.20$ resp – not bad!

`¿Run experiments on Pomerance's problem directly?`

Methods used

First and second moment methods from probabilistic combinatorics.

Husimi graphs from statistical physics.

Lagrange inversion from algebraic combinatorics.

Analytic continuation of solutions to functional equations.

Comparative estimates on smooth numbers, via comparative estimates on saddle points.

Random graph theory, Poisson processes and conversion from continuous to discrete.