

## Curves of genus 2 with many rational points via K3 surfaces

Noam D. Elkies<sup>1</sup>

Dept. of Mathematics, Harvard University, Cambridge, MA 02138  
(elkies@math.harvard.edu)

**Abstract.** Let  $C$  be a (smooth, projective, absolutely irreducible) curve of genus  $g \geq 2$  over a number field  $K$ . Faltings [Fa1,Fa2] proved that the set  $C(K)$  of  $K$ -rational points of  $C$  is finite, as conjectured by Mordell. The proof can even yield an effective upper bound on the size  $\#C(K)$  of this set (though not, in general, a provably complete list of points); but this bound depends on the arithmetic of  $C$ . This suggests the question of how  $\#C(K)$  behaves as  $C$  varies. Following [CHM], we define for each  $g \geq 2$  and  $K$ :

$$B(g, K) = \max_C \#C(K),$$

with  $C$  running over all curves over  $K$  of genus  $g$ ;

$$N(g, K) = \limsup_C \#C(K) [\leq B(g, K)],$$

(so infinitely many  $C$  have  $N$  rational points over  $K$ , but only finitely many have more than  $N$ ); and<sup>2</sup>  $N(g) = \max_K N(g, K)$ . It is not known whether either  $B(g, K)$  or  $N(g)$  is finite for any  $g, K$ ; even the question of whether  $N(2, \mathbf{Q}) < \infty$  is very much open. Caporaso, Harris and Mazur proved [CHM] that Lang's Diophantine conjectures [La] imply the finiteness of  $B(g, K)$ ,  $N(g)$  for any number field  $K$  and integer  $g \geq 2$ ; but the proof yields no estimates on these bounds.

While giving upper bounds seems hopeless at present, lower bounds are more tractable: we need only construct curves or families of curves with many rational points. We announce several new constructions, all using K3 surfaces of maximal Picard number. Specifically, we begin with the K3 surface  $S/\mathbf{Q}$  whose Néron–Severi group has rank 20 and discriminant  $-163$  and consists of divisor classes defined over  $\mathbf{Q}$ .<sup>3</sup> We use models of  $S$  as the double cover  $W^2 = P_6(X, Y, Z)$  of  $\mathbf{P}^2$  branched along a sextic curve  $C_6 : P_6(X, Y, Z) = 0$ . There is a finite but large number (50+) of lines  $l_i : \lambda_i(X, Y, Z) = 0$  on which  $P_6$  restricts to a perfect square (geometrically these are the tritangent lines of  $C_6$ ). The restriction  $P_6|_L$  of  $P_6$  to a generic line  $L \subset \mathbf{P}^2$  thus yields a genus-2 curve  $C_L : w^2 = P_6|_L$  with a pair of rational points above the intersection of  $L$  with each  $l_i$ .

More detailed study of the configuration of the lines  $l_i$ , and of rational curves of higher degree in  $\mathbf{P}^2$  on which  $P_6$  restricts to a perfect square, leads to the following lower bounds.

<sup>1</sup> Supported in part by NSF grant DMS-0501029

<sup>2</sup> It is essential to use  $N(g, K)$  here rather than  $B(g, K)$ , because even for a single curve  $C$  over a number field  $K_0$  it is clear that by enlarging  $K \supset K_0$  we can make  $\#C(K)$  arbitrarily large.

<sup>3</sup> See [E1, p.9] for a model of  $S$  as an elliptic K3 surface with Mordell–Weil group  $(\mathbf{Z}/4\mathbf{Z}) \oplus \mathbf{Z}^4$ .

**Theorem 1.** *There exist infinitely many genus-2 curves over  $\mathbf{Q}$  with at least 75 pairs of rational points. Thus  $N(2, \mathbf{Q}) \geq 150$ .*

The previous bound was Mestre's  $N(2, \mathbf{Q}) \geq 48$ ; Mestre's curves have 12 automorphisms, whereas our curves have no automorphisms other than the identity and the hyperelliptic involution.

**Theorem 2.** *There exist infinitely many genus-2 curves over  $\mathbf{Q}$  with a rational Weierstrass point and at least 59 pairs of rational points, for a total of  $\geq 119$ .*

Equivalently, there are infinitely many quintics  $P_5$  without repeated roots such that the Diophantine equation  $y^2 = P_5(x)$  has at least 119 rational solutions. We do not know of a previous record for such equations, but it would surely not exceed Mestre's 48 for an unrestricted genus-2 curve.

By varying  $L$  we have searched for individual genus-2 curves with even more rational points than promised by Theorems 1 and 2, using M. Stoll's program `ratpoints`. We did not succeed in improving Kulesz and Keller's lower bound of 588 on  $B(2, \mathbf{Q})$  [KK]. But their curve, like Mestre's curves, has 12 automorphisms, so its 588 points fall into "only" 49 orbits under the automorphism group. For genus-2 curves with minimal automorphism group, the record was 366 [St]. We raise this to at least 536 for the curve

$$y^2 = 2380^2x^6 + 947240x^5 - 29926671x^4 + 6414496x^3 + 46164876x^2 - 1258740x + 420^2.$$

For curves  $y^2 = P_5(x)$ , we find 347 points on

$$y^2 = 372468096x^5 - 830776095x^4 + 607949578x^3 - 108403791x^2 - 49652776x + 4028^2.$$

The degree-zero divisors on  $C_L$  supported on the known rational points generate a subgroup of  $\text{Jac}(C_L)$  of generic rank 18. We find infinitely many specializations of rank at least 21 over  $\mathbf{Q}$ , and two of rank at least 26, one of which is given by

$$y^2 = 80878009x^6 - 236558406x^5 - 1018244179x^4 + 4436648480x^3 + 6445563464x^2 - 13620761544x + 68406^2.$$

This improves on Stahlke's record of 22 [St] for an absolutely simple genus-2 Jacobian over  $\mathbf{Q}$ .

## References

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