Institut de Mathématiques de Luminy

Computing the 2-distribution of points on Hermitian surfaces (ANTS-8, Banff, CA)

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CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Weight Distribution

Definition 1

The h-distribution of points of a projective variety $\mathcal{V}$ is the decreasing sequence of the number of points in the section of $\mathcal{V}$ by all hypersurfaces of degree h.

- Let $X \subset \mathbb{P}^m(\mathbb{F}_q)$ and $N = \#X(\mathbb{F}_q)$

$$c : \mathcal{F}_h(V, \mathbb{F}_q) \longrightarrow \mathbb{F}_q^N$$

$$f \longmapsto c(f) = (f(P_1), \ldots, f(P_N))$$

$C_h(X) = \text{Im} \quad c$

- **definition** Let $c(f)$ be a codeword

$$cw(f) = \#\{P \in X \mid \quad f(P) = 0\}$$

$$w(c(f)) = \#X(\mathbb{F}_q) - cw(f)$$

$$\text{dist}C_h(X) = \#X(\mathbb{F}_q) - \max_{f \in \mathcal{F}_h} cw(f)$$

- **Proposition** The parameters of $C_h(X)$:
  lenght $C_h(X) = \#X(\mathbb{F}_q)$,

$$\dim C_h(X) = \dim \mathcal{F}_h - \dim \ker c,$$

$$\text{dist}C_h(X) = \#X(\mathbb{F}_q) - \max_{f \in \mathcal{F}_h} \#X_{Z(f)}(\mathbb{F}_q)$$

If c injective $\Rightarrow \dim C_h(X) = \binom{m+h}{h}$

The study of $C_2(X)$ over F_q

$X : x_0^{t+1} + x_1^{t+1} + x_2^{t+1} + x_3^{t+1} = 0$

- 1. Number of points of X

$\#X(\mathbb{F}_q) = (t^2 + 1)(t^3 + 1)$, 1966

- 2. Injectivity of the application c

Tsfasman-Serre-Sørensen Bound $\Rightarrow$ $h \leq t$.

- 3. History of $C_h(X)$

$h = 2, t = 2$ { R. Tobias, 1985 ; P. Spurr, 1986 }

$h = 2, t = 2$ A. B. Sørensen, 1991

Conjecture: $\#X_{Z(f)}(\mathbb{F}_q) \leq h(t^3 + t^2 - t) + t + 1$

G. Lachaud, A.G.C.T-4, 1993

$\#X_{Z(f)}(\mathbb{F}_q) \leq h(t^3 + t^2 + t + 1)$

The weight distribution over F_4

- The code $C_2(X)$ defined on $\mathbb{F}_4$ is
  $[45, 10, 22]_4$-code. And it is a even-weight code.
  We have the following formula:

$$w_i = (10 + i) \times 2 \quad i = 1, ..., 12$$

$A_{w_1} = 2.160$, $A_{w_2} = 2.970, A_{w_3} = 4.320,$
$A_{w_4} = 40.500, A_{w_5} = 122.976$, $A_{w_6} = 233.415, A_{w_7} = 285.120, A_{w_8} = 233.400,$
$A_{w_9} = 97.200, A_{w_{10}} = 20.574, A_{w_{11}} = 4.320, A_{w_{12}} = 1.620$

Resolution of Sørensen Conjecture ($h \leq 2$ and $t = p^a$)

Definition 2

For any projective algebraic variety $\mathcal{V}$, the maximum dimension $g(\mathcal{V})$ of linear subspaces lying on $\mathcal{V}$, is called the projective index of $\mathcal{V}$. The largest dimensional spaces contained in $\mathcal{V}$ are called the generators of $\mathcal{V}$.

For $\mathcal{Q}$ a degenerate quadric and r($\mathcal{Q}$)=r, $\mathcal{Q}$ is a cone $\Pi_{n-r}\mathcal{Q}_{r-1}$ with vertex Π_{n-r} (the set of singular points of $\mathcal{Q}$) and base $\mathcal{Q}_{r-1}$ in a subspace Π_{r-1} skew to Π_{n-r}.

Definition 3

For $\mathcal{Q} = \Pi_{n-r}\mathcal{Q}_{r-1}$ a degenerate quadric with $r(\mathcal{Q}) = r$, $\mathcal{Q}_{r-1}$ is called the non-degenerate quadric associated to $\mathcal{Q}$. The degenerate quadric $\mathcal{Q}$ will be said to be of hyperbolic type (resp. elliptic, parabolic) if its associated non-degenerate quadric is of that type.

Table 1: Quadrics in PG(3,q)

r($\mathcal{Q}$)	Description	$\lvert\mathcal{Q}\rvert$	g($\mathcal{Q}$)
1	repeated plane $\Pi_2\mathcal{P}_0$	π_2	2
2	pair of distinct planes $\Pi_2\mathcal{H}_1$	$2q^2 + \pi_1$	2
2	line $\Pi_1\mathcal{E}_1$	π_1	1
3	cone quadric $\Pi_0\mathcal{P}_2$	π_2	1
4	hyperbolic quadric $\mathcal{H}_3(\mathcal{R}, \mathcal{R}')$	$\pi_2 + q$	1
4	elliptic quadric $\mathcal{E}_3$	$\pi_2 - q$	0

Some values of $\#X_{Z(f)}(\mathbb{F}_q)$

$s(t) = 2t^3 + 2t^2 - t + 1$,
$s_2(t) = 2t^3 + t^2 + 1$,
$s_3(t) = 2t^3 + t^2 - t + 1$,
$s_4(t) = 2t^3 + 1$, $s_5(t) = 2t^3 - t + 1$

The methods used

- 1. Hyperplane section of non-singular Hermitian variety.
- 2. The trace on a hyperplane of the two varieties and a special covering of PG(3,q).
- 3. The structure of two reguli, each one generating the hyperbolic quadric and the upper bound on the number of skew lines contained in a regulus and the non-singular Hermitian surface.
- 4. The geometrical structure of a cone quadric of rank 3 which is in fact a union of $q + 1$ lines through a vertex and the upper bound on the number of common lines of the two varieties.

The Weight Distribution (w_i, A_{w_i}) of $C_2(X)$ ($\mathbb{F}_{t^2}$)

$w_1 = t^5 - t^3 - t^2 + t$
The codewords << w_1 >>: union of 2 tan planes to X and $l \cap X = (t+1)$ points.
$A_{w_1} = (t^2 - 1)[\frac{1}{2}(t^5 + t^3 + t^2 + 1)t^5]$

$w_2 = t^5 - t^3$
The codewords << w_2 >> are given by:
–hyperbolic containning III of X.
–union of 2 planes tan of X and $l \subset X$.
–union of 2 planes one tan, the second n-tan to X and $l \cap X = 1$ point.
$A_{w_2} = (t^2 - 1)[\frac{1}{2}(t^5 + t^3 + t^2 + 1)(3t^2 - t + 1)t^2]$

$w_3 = t^5 - t^3 + t$
The codewords << w_3 >>: quadrics which are union of 2 planes one tan, the second non-tan to X and $l \cap X = (t+1)$ points.
$A_{w_3} = (t^2 - 1)(t^5 + t^3 + t^2 + 1)(t^6 - t^5)$

Conjecture on w_4 and w_5

- $w_4 = t^5 - t^3 + t^2$
  The codewords << w_4 >> are given by quadrics which are union of 2 planes tan to X and $l \cap X = 1$ point, and particular $\mathcal{E}_3$.
- $w_5 = t^5 - t^3 + t^2 + t$
  The codewords << w_5 >>:union of 2 planes non-tan to X and $l \cap X = (t+1)$ pts, and another particular $\mathcal{E}_3$.

Divisibility by t of the others weights ? (**Theorem of Ax (1964)**)

Table 2: Quadratic section of X

Rank	Type	$\#X_{Z(\mathcal{Q})}(\mathbb{F}_q)$	$\#X_{Z(\mathcal{Q})}(\mathbb{F}_4)$	Weight over $\mathbb{F}_q$	Weight over $\mathbb{F}_4$
1 (repeated plane)	1	$t^3 + t^2 + 1$	13	t^5	32
	2	$t^3 + 1$	9	$t^5 + t^2$	36
2 (line)	3	1	1	$t^5 + t^3 + t^2$	44
	4	$t + 1$	3	$t^5 + t^3 + t^2 - t$	42
	5	$t^2 + 1$	5	$t^5 + t^3$	40
2 (pair of distinct planes) $\mathcal{Q} = \mathcal{P}_1 \cup \mathcal{P}_2$	6	$s_4(t)$	17	$t^5 - t^3 + t^2$	28
		$s_5(t)$	15	$t^5 - t^3 + t^2 + t$	30
	7	$s_3(t)$	19	$t^5 - t^3 + t$	26
		$s_2(t)$	21	$t^5 - t^3$	24
	8	$s(t)$	23	$t^5 - t^3 - t^2 + t$	22
		$s_2(t)$	21	$t^5 - t^3$	24
3 (cone)	9	$\leq t^3 + t^2 + t$ $+1 < s_4(t)$	≤ 15	$\geq t^5 - t$	$30 \leq w \leq 44$
	10	$t^3 + t^2 + 1$	13	t^5	32
		$t^3 + 2t^2 - t + 1$	15	$t^5 - t^2 + t$	30
4 (hyperbolic) $\mathcal{H}(\mathcal{R}, \mathcal{R}')$	11	$s_2(t)$	21	$t^5 - t^3$	24
	12	$\leq t^3 + 3t^2 - t$ $+1 \leq s_3(t)$	≤ 19	$\geq t^5 - t^3$	$26 \leq w \leq 32$
	13	$\leq t^3 + 2t^2$ $+1 \leq s_4(t)$	≤ 17	$\geq t^5 - t^2$	$28 \leq w \leq 36$
	14	$\leq t^3 + t^2 +$ $t + 1 < s_4(t)$	≤ 15	$\geq t^5 - t$	$30 \leq w \leq 40$
4 (elliptic)	15	$\leq 2t^3 + 2t$ $+2 < s_2(t)$	≤ 17	$\geq t^5 - t^3 + t^2$ $-2t - 1$	$28 \leq w \leq 45$

Resolution of Sørensen Conjecture ($h \geq 3$ and $t = p^a$)