Non-vanishing of Dirichlet L-functions at the Central Point

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The complex number s is denoted by $s = \sigma + it$

Definition (A. Selberg 1989)

The Selberg Class consists of functions of complex variable F that satisfy the following conditions :

i) Dirichlet Series. For $\sigma > 1$, $F(s)$ can be written $F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}$, where $a(1) = 1$.

ii) Analytic Continuation. There exists an integer $m \geq 0$ such that $(s-1)^m F(s)$ has an analytic continuation of finite order.

iii) Functional Equation . There exist real numbers $Q > 0$, $\lambda_j > 0$ and for $1 \leq j \leq r$, complex numbers μ_j with $Re(\mu_j) \geq 0$ such that the function

$$\phi(s) = Q^s \left(\prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)\right) F(s) = \gamma(s) F(s)$$

satisfies the functional equation

$$\phi(s) = \omega \overline{\phi}(1-s),$$

where $\omega \in \mathbb{C}$, $|\omega| = 1$ and $\overline{\phi}(s) = \overline{\phi(\overline{s})}$.
iv) Eulerian product. The function F can be written

$$F(s) = \prod_p F_p(s),$$

where $F_p(s) = exp\left(\sum_{k=1}^{+\infty} \frac{b(p^k)}{p^{ks}}\right)$ and $b(p^k) = O(p^{k\theta})$ for a certain $\theta < \frac{1}{2}$ and p is a prime number.
v) Ramanujan Hypothesis. For any fixed $\epsilon > 0$, we have as n is enough large $a(n) = O(n^{\epsilon})$.

For any function F in the Selberg class S, we define the degree of F as :

$$d = d_F = 2 \sum_{j=1}^{r} \lambda_j .$$

- We denote by S_d the set of functions of S with fixed degree d.
- We denote by $S_d^{\#}$ the set of functions of $S^{\#}$ with fixed degree d.

Remark :

- $(S, .)$ is a monoid.
- If $F \in S$ is entire and if $\theta \in \mathbb{R}$ then $F_\theta(s) = F(s+i\theta)$ belongs to S (This holds also for $S^{\#}$).

Theorem. (Conrey and Ghosh 1993)
If $F \in S$ then $F_p(s)$ does not have zeros in $\sigma > 1$. Therefore F does not vanish for $\sigma > 1$.

The zeros of a function in S are :
- **The trivial zeros**

$$-\frac{n+\mu_j}{\lambda_j}\text{ , where } n \in \mathbb{N} \text{ and } 1 \leq j \leq r.$$

- **The non-trivial zeros** in the critical strip $0 \leq \sigma \leq 1$.

Examples :

-The Riemann zeta function is defined for $\sigma >$ 1 by :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}.$$

It has the functional equation

$$\phi(s) = \phi(1 - s),$$

where

$$\phi(s) = \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s),$$

where Γ is the gamma function.

-The Dirichlet L-functions is defined for $\sigma > 1$ by :

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where χ is a primitive Dirichlet character (mod q). It also satisfies the functional equation

$$\phi(s,\chi) = \frac{\tau(\chi)}{i^a\sqrt{q}}\phi(1-s,\overline{\chi}),$$

where

$$\phi(s) = \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma(\frac{s}{2}+\delta)L(s,\chi),$$

$\tau(\chi) = \sum_{n(mod\ q)} \chi(n)e^{\frac{2i\pi n}{q}}$, $\delta = 0$ if $\chi(-1) = 1$ and $\delta = 1$ if $\chi(-1) = -1$.

-The Dedekind zeta function is defined for $\sigma > 1$ by :

$$\zeta_k(s) = \sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s} = \prod_p (1 - Np^{-s})^{-1},$$

where k is a number field, $\mathfrak{a}$ runs over the ideals of k and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. It satisfies the functional equation :

$$\phi(s) = \phi(1-s),$$

where

$$\phi(s) = \left(\frac{\sqrt{|d_k|}}{2^{r_2} \pi^{\frac{n}{2}}} \right)^s \Gamma(\tfrac{s}{2})^{r_1} \Gamma(s)^{r_2} \zeta_k(s),$$

where d_k is the discriminant of k, $n = [k, \mathbb{Q}]$ is the degree of the extension $k \mid \mathbb{Q}$, r_1 is the number of real places of k and r_2 is the number of complex conjugates of k.

-The Elliptic Curves

Let E be an elliptic curve defined over $\mathbb{Q}$ by the minimal model :

$$y^2 + k_1xy + k_2y = x^3 + k_3x^2 + k_4x + k_5,$$

where $k_i \in \mathbb{Z}$, $(1 \leq i \leq 5)$ and N is its conductor. We define

$$L_E(s) = \prod_{p|N}(1-a_pp^{-\frac{1}{2}-s})^{-1}\prod_{p\nmid N}(1-a_pp^{-\frac{1}{2}-s}+p^{-2s})^{-1},$$

where $a_p = p + 1 - \text{card}(E_p)$ if $(p \nmid N)$.
By a theorem of Hasse, we have $a_p < 2\sqrt{p}$. Therefore $L_E(s)$ is convergent for $\mathcal{R}e(s) > 1$ and the following functional equation can be derived as follows :

$$\Lambda_E(s) = C\Lambda_E(1-s),$$

where

$$\Lambda_E(s) = (\frac{\sqrt{N}}{2\pi})^s\Gamma(s+\frac{1}{2})L_E(s).$$

Conjectures :

• The Riemann Hypothesis : All the non-trivial zeros lie on the line $\Re e(s) = \frac{1}{2}$: They are also simple

• The Selberg Othonormality Conjecture (SOC) :

$$\sum_{p \leq x} \frac{a_F(p)\overline{a_G(p)}}{p} = \delta_{F,G} log_2 x + O(1),$$

where $\delta_{F,G} = 1$ if $F = G$ and 0 otherwise.

• The Chowla Conjecture : $L(1/2, \chi) \neq 0$ for primitive Dirichlet characters χ.

• Serre extended the Chowla Conjecture to Artin L-functions of irreducible characters of Galois extentions over $\mathbb{Q}$.

Results :

• R. Balasubramanian, K. Murty 1992 :
true for 4 % of characters mod q.

• H. Iwaniec, P. Sarnak 1999 :
true for 1/3 of characters mod q.

• R. Murty 1989 (GRH) :
true for 50 % of characters mod q.

• E. Özluk, P. Snyder 1999 (GRH) :
true for 15/16 of fundamental discriminants d, $L(1/2, \chi_d) \neq 0$.

• K. Soundararajan 2000 :
true for $87,5\%$ of odd $d > 0$, $L(1/2, \chi_{8d}) \neq 0$.

• N. Katz, P. Sarnak 1999 : (Under the "Pair-Correlation" Conjecture for "small" zeros of L functions) : true for 100 % of $L(1/2, \chi_d) \neq 0$.

Computational Results :

- M. Watkins 2004 :

$L(s, \chi)$ has no real zeros for real odd χ of modulus $d \leq 3 \times 10^8$. The last record was up to 3×10^5 due to Low and Purdue (1968).

- Kok Seng 2005 :

$L(s, \chi)$ has no real zeros for real even χ of modulus $d \leq 2 \times 10^5$. The last record was up to 986 due to Rosser.

- Explicit computation of values of L-functions require $O(\sqrt{q})$ terms by using the smooth approximate functional equation : R. Rumely (93), A. Odlyzko(79), E. Tollis(98), M. Watkins (04), T. Dokchitser(04), M. Rubinstein (01) .

- Computing zeros of L-functions using the Explicit Formula in Number Theory : S. Omar (01, 08), A. Booker (06, 07), D. Miller (02).

Theorem (Explicit Formula)

Let F satisfy $F(0) = 1$ together with the following conditions :

(A) F is even, continuous and continuously differentiable everywhere except at a finite number of points a_i, where $F(x)$ and $F'(x)$ have only a discontinuity of the first kind, such that $F(a_i) = \frac{1}{2}(F(a_i + 0) + F(a_i - 0))$.

(B) There exists a number $b > 0$ such that $F(x)$ and $F'(x)$ are $O(e^{-(\frac{1}{2}+b)|x|})$ as $\mid x \mid \to \infty$.
Then the Mellin transform of F :

$$\Phi(s) = \int_{-\infty}^{+\infty} F(x) e^{(s-\frac{1}{2})x} dx$$

is holomorphic in every vertical strip
$-a \leq \sigma \leq 1 + a$ where $0 < a < b,\ a < 1$

We have :

$$\sum_{\rho} \Phi(\rho) \quad = \quad \ln\left(\frac{q}{\pi}\right) - I_\delta(F) - 2\sum_{p,m\geq 1} \mathcal{R}e(\chi^m(p))F(m\ln(p))\frac{\ln(p)}{p^{m/2}}$$

where

$$I_\delta(F) = \int_0^{+\infty} \left(\frac{F(x/2)e^{-(\frac{1}{4}+\frac{\delta}{2})x}}{1-e^{-x}} - \frac{e^{-x}}{x} \right) dx,$$

and

$$\delta = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

Conditional Bounds

We denote by γ_k the imaginary part of the k^{th} zéro of $L(s,\chi)$ and $n_\chi = \text{ord} L(1/2,\chi)$.
Let

$$S(y) = n_\chi + \sum_{k\neq 0} n_k e^{-\gamma_k^2/4y}.$$

By the explicit formula, we have the identity

$$S(y) = \sqrt{\frac{y}{\pi}}\left(\ln(\frac{q}{\pi}) - I_\delta(F_y) - 2\sum_{p,m}\frac{\ln(p)}{p^{m/2}}\mathcal{R}e(\chi^m(p))e^{-y(m\ln(p))^2}\right).$$

Proposition

Assuming GRH, we have for all $y > 0$

$$n_\chi \leq S(y)$$

and

$$\lim_{y\to 0} S(y) = n_\chi.$$

Unconditional Bounds

We would have $\Re e\,\Phi(s) \geq 0$ for $0 \leq \sigma \leq 1$. Then, we consider $G_y(x) = \frac{F_y(x)}{\cosh(x/2)}$ where F_y is the conditional test function. Actually on both lines $\sigma = 0, 1$, $\Re e\,\Phi(\sigma + it) = \widehat{F_y}(t) \geq 0$.

We set

$$T(y) = n_\chi + \frac{1}{2\int_0^{+\infty} \frac{e^{-yx^2}}{\cosh(x/2)}\,dx} \sum_{\rho \neq \frac{1}{2}} \Re e\,\Phi(\rho).$$

By the Explicit Formulas, we have the identity

$$T(y) \quad = \quad g_y \left(\ln(\frac{q}{\pi}) - I_\delta(G_y) - 4 \sum_{p,m} \frac{\ln p}{1 + p^m} \mathcal{R}e(\chi^m(p)) e^{-y(m \ln p)^2} \right)$$

where

$$g_y = \frac{1}{2\int_0^{+\infty} \frac{e^{-yx^2}}{\cosh(x/2)}\,dx}$$

Proposition

For all $y > 0$, we have

$$n_\chi \leq T(y)$$

and

$$\lim_{y \to 0} T(y) = n_\chi.$$

Proposition

Under GRH

$L(\frac{1}{2}, \chi) \neq 0$ holds if and only if there exists $y > 0$ such that $S(y) < 1$.

Proposition

$L(\frac{1}{2}, \chi) \neq 0$ holds if and only if there exists $y > 0$ such that $T(y) < 1$.

Theorem

Under GRH, we have

$$n_\chi \ll \frac{\ln(q)}{\ln\ln(q)},$$

Unconditionally, we have

$$n_\chi < \ln(q).$$

Theorem

Let $\rho_\chi = 1/2 + i\gamma_\chi$ be the lowest zero on the critical line of $L(s,\chi)$.

Under GRH, we have

$$|\gamma_\chi| \ll \frac{1}{\ln\ln(q)}$$

These estimates remain true for the Selberg Class in term of the conductor.

Numerical Evidence for $n_\chi = 0$

$\mathbf{q}$	$\mathbf{y_0}$	$\mathbf{max_\chi S(y_0)}$	$\mathbf{y}$	$\mathbf{max_\chi T(y)}$	p_0	$\mathbf{n_\chi}$	*time*
$10 \leq q < 10^2$	0.3	0.50410	0.3	0.67812	100	0	$50m$
$10^2 \leq q < 10^3$	0.2	0.46543	0.2	0.57037	10^3	0	$14h$
$10^3 \leq q < 10^4$	0.11	0.41720	0.11	0.52140	4×10^3	0	$4\ d$
$10^4 \leq q < 10^5$	0.09	0.34512	0.09	0.37528	10^4	0	$6\ d$
$10^5 \leq q < 10^6$	0.08	0.28643	0.07	0.31726	6×10^4	0	$9\ d$
$10^6 \leq q < 10^7$	0.07	0.13242	0.07	0.09642	10^5	0	$14\ d$
$10^7 \leq q < 10^8$	0.05	0.07830	0.05	0.08347	5×10^5	0	$20\ d$
$10^8 \leq q < 10^9$	0.04	0.05176	0.04	0.06941	10^6	0	$50\ d$
$10^9 \leq q < 10^{10}$	0.01	0.25871	0.01	0.35762	5×10^6	0	$180\ d$

Theorem (The Li Criterion 1997)

$$(RH) \Leftrightarrow \lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}\left[s^{n-1}\log\xi(s)\right]_{s=1} \geq 0, \forall n \in \mathbb{N},$$

where $\quad \xi(s) = s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s).$

Also, we have

$$\lambda_n = \sum_{\rho}\left[1 - (1 - \frac{1}{\rho})^n\right],$$

where ρ runs over the set of the non-trivial zeros of ζ.

The Generalized Li Criterion

In 1999, Bombieri and Lagarias proved the following theorem :

Let $\mathfrak{R}$ be a set of complex numbers ρ of multiplicity a positive integer and such that $1 \notin \mathfrak{R}$ and $\sum_{\rho} \frac{1 + |Re(\rho)|}{(1 + |\rho|)^2} < \infty$ then the following conditions are equivalent :

1- $Re\rho \leq \frac{1}{2}$ for any $\rho \in \mathfrak{R}$.

2- $\sum_{\rho} Re \left[1 - \left(1 - \frac{1}{\rho} \right)^{-n} \right] \geq 0$ for $n = 1, 2, ...$

Corollarly (Li's Generalized Criterion).

For any $F \in S$, we have

$$(H\ \ R\ G) \Leftrightarrow \lambda_F(n) = \sum_{\rho} \left[1 - (1 - \frac{1}{\rho})^n\right] \geq 0,\ \forall\ n \in \mathbb{N},$$

where ρ is a non-trivial zero of F and

$$\sum_{\rho} = \lim_{T \mapsto +\infty} \sum_{|Im(\rho)|<T}.$$

Theorem (S.Omar, K. Mazouda 2007)

Let F be a function in the Selberg class. We assume that F does not vanish on the ligne $Re(s) = 1$. Then, we have

$$\begin{aligned} \lambda_F(n) &= m_F + n(logQ_F - \frac{d_F}{2}\gamma) \\ &- \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} \lim_{X \to +\infty} \left\{ \sum_{k \le X} \frac{\Lambda(k)}{k} (logk)^{l-1} - \frac{m_F}{l} (\log X)^l \right\} \\ &+ n \sum_{j=1}^{r} \lambda_j \left(-\frac{1}{\lambda_j + \mu_j} + \sum_{l=1}^{+\infty} \frac{\lambda_j + \mu_j}{l(l + \lambda_j + \mu_j)} \right) \\ &- \sum_{j=1}^{r} \sum_{k=2}^{n} \binom{n}{k} (-\lambda_j)^k \sum_{l=1}^{+\infty} \left\{ \frac{1}{l + \lambda_j + \mu_j} \right)^k . \end{aligned}$$

Theorem (S.Omar, K.Mazouda 2008)

- Under (GRH), we have :

$$\lambda_F(n) = \frac{d_F}{2} n \log n + c_F n + O\left(\sqrt{n} \log n\right),$$

with

$$c_F = \frac{d_F}{2}(\gamma - 1) + \frac{1}{2} \log(\lambda Q_F^2), \lambda = \prod_{j=1}^{r} \lambda_j^{2\lambda_j}.$$

- If $\lambda_F(2) > 0$ then there does not exist a siegel zero for $F(s)$.