Supplementary Tables for "Numerical Results on Class Groups of Imaginary Quadratic Fields"

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We present supplemental tables and additional data that extend that presented in [7]. Data corresponding to all the conjectures mentioned in [7] are included, and all tables are complete, including previously published results. In addition, two corrections to the data in [7] are included:

- Originally, we only listed first occurrences of p-Sylow subgroups for primes $p \leq 173$. In this paper, we present the entire list, for primes $p \leq 389$. See Table 7.
- When listing the first Δ needing prime ideals of norm up to p, we pointed out an anomaly in the data at $p = 181$. Subsequent analysis has shown this to be a bug in our statistics gathering program. The data no longer contains any anomalies of this sort. See Table 15.

Bounds on $L(1, \chi)$

There has been significant interest [2, 3, 6, 11] in the extreme values of $L(1, \chi_\Delta)$ due to the relationship between it and the class number h_Δ. This can be seen in the analytic class number formula,

$$L(1, \chi_\Delta) = \frac{h_\Delta \pi}{\sqrt{|\Delta|}} \ ,$$

where extreme values of $L(1, \chi_\Delta)$ correspond to extreme values of h_Δ.

In [10], Littlewood developed bounds on $L(1, \chi_\Delta)$, namely that under the ERH,

$$\{1 + o(1)\}(c_1 \log\log \Delta)^{-1} < L(1, \chi_\Delta) < \{1 + o(1)\} c_2 \log\log(\Delta) \ , \tag{0.1}$$

where c_1 and c_2 are defined as follows:

$$c_1 = 12e^\gamma/\pi^2 \text{ and } c_2 = 2e^\gamma \text{ when } 2 \nmid \Delta$$
$$c_1 = 8e^\gamma/\pi^2 \text{ and } c_2 = e^\gamma \text{ when } 2 \mid \Delta \ .$$

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In [11], Shanks investigated Littlewood's bounds, and defined two values he termed the *upper* and *lower Littlewood indices*

$$\begin{aligned} ULI &= L(1,\chi_\Delta)/(c_2 \log\log \Delta) \\ LLI &= L(1,\chi_\Delta) c_1 \log\log \Delta \ . \end{aligned}$$

These indices effectively ignore the $o(1)$ given in Littlewood's bounds. We would expect extreme values of the LLI and the ULI to approach 1.

Finally, as in [11], we define the function

$$L_\Delta(1) = \prod_{p \text{ prime}} \left(\frac{p}{p - \left(\frac{4\Delta}{p}\right)} \right) \ ,$$

which is essentially $L(1,\chi_\Delta)$ with the 2-factor divided out. Shanks defines bounds on $L_\Delta(1)$ similar to (0.1)

$$\{1+o(1)\} \left(\frac{8}{\pi^2} \log\log 4\Delta \right)^{-1} < L_\Delta(1) < \{1+o(1)\} e^\gamma \log\log 4\Delta \ ,$$

and the corresponding indices

$$\begin{aligned} ULI_\Delta &= L_\Delta(1)/\left(e^\gamma \log\log 4\Delta\right) \\ LLI_\Delta &= L_\Delta(1) \frac{8}{\pi^2} \log\log 4\Delta \ . \end{aligned}$$

In order to test the validity of these conditional bounds, we recorded successive minimum and maximum values, and corresponding ULI and LLI values, of $L(1,\chi_\Delta)$ for discriminants Δ, with $\Delta \equiv 0 \pmod 4$, $\Delta \equiv 1 \pmod 8$ and $\Delta \equiv 5 \pmod 8$. The maximum $L(1,\chi_\Delta)$ found was 8.09414... ($ULI = 0.70996$) for $\Delta = -45716419031$. The maximum ULI value was 0.73202... ($L(1,\chi_\Delta) = 4.14624...$) for $\Delta = -27867502724$. The minimum $L(1,\chi_\Delta)$ found was 0.17448... ($LLI = 1.2188...$) for $\Delta = -8570250280$. The minimum LLI value was 1.10314... ($L(1,\chi_\Delta) = 0.39502...$) for $\Delta = -1012$.

In Table 1 we list successive maximum $L(1,\chi_\Delta)$ and corresponding ULI values with $\Delta \equiv 1 \pmod 8$, as the values in this congruence class are the overall maximum. In Table 2 we list successive minimum $L(1,\chi_\Delta)$ and corresponding LLI values with $\Delta \equiv 5 \pmod 8$, as the values in this congruence class are the overall minimum. The $L(1,\chi_\Delta)$ values correspond to Buell's previous tabulations [3] and so we only display the maximum and minimum values which follow after Buell's data.

Following Buell, we also calculated the mean values of $L(1,\chi_\Delta)$ for discriminants $\Delta \equiv 0 \pmod 4$ and $\Delta \equiv 1 \pmod 4$. These values, 1.18639... and 1.58185... are similar to Buell's findings [3].

The Cohen-Lenstra Heuristics

In [5], Cohen and Lenstra presented a number of heuristics regarding class groups of quadratic number fields. During our computations, we tested the frequency with which odd primes p divide the class number h_Δ, the frequency that the odd part of the class group is non-cyclic, and the number of non-cyclic factors of the p-Sylow subgroups.

Divisibility of h_Δ by Odd Primes. For an imaginary quadratic field with discriminant Δ, the probability that an odd prime p divides the class number h_Δ is conjectured in [5] as

$$\mathrm{prob}(p \mid h_\Delta) = 1 - \eta_\infty(p) \ , \tag{0.2}$$

where $\eta_\infty(p) = \prod_{k\geq 1} 1-p^{-k}$. As observed by Buell [3], under the same heuristic assumptions, p^2 divides the class number h_Δ with probability $1 - \frac{p\eta_\infty(p)}{p-1}$ and p^3 divides the class number with probability $1 - \frac{p^3\eta_\infty(p)}{(p-1)^2(p+1)}$. We define the value $p_l(x)$ as the observed ratio of discriminants less than x with $l \mid h_\Delta$ divided by the conjectured probability shown in (0.2). As x increases, we would expect the value of $p_l(x)$ to approach 1. Similarly, we define the ratios $p_{l^2}(x)$ for l^2 dividing the class number, and $p_{l^3}(x)$ for l^3 dividing the class number.

In Table 3, we present the values of $p_l(x)$ for small primes l. The values appear to approach 1 from below. The values of $p_{l^2}(x)$ and $p_{l^3}(x)$ approach 1 from below in a similar fashion, and so are not presented here. It should be noted that the ratios approach 1 at a slower rate for l^2 and an even slower rate for l^3.

Cyclic Cl^*_Δ. Define Cl^*_Δ to be the odd part of Cl_Δ. The heuristics given in [5] state that the probability that Cl^*_Δ is cyclic is equal to

$$\mathrm{prob}(Cl^*_\Delta \text{ cyclic}) = \frac{\zeta(2)\zeta(3)}{3\zeta(6)C_\infty\eta_\infty(2)} \ , \tag{0.3}$$

where $C_\infty = \prod_{i\geq 2}\zeta(i)$. This value is roughly 97.7575%. We define $c(x)$ as the observed ratio of discriminants less than x with Cl^*_Δ cyclic divided by the conjectured probability shown in (0.3). As x increases, we would expect the value of $c(x)$ to approach 1.

In Table 4, we present values of $c(x)$, along with the total number of discriminants less than x with Cl^*_Δ non-cyclic. As expected, the values of $c(x)$ approach 1 from above.

Non-Cyclic Factors of p-Sylow Subgroups. For an odd prime p, define the p-rank of Cl_Δ as the number of non-cyclic factors of the p-Sylow subgroup of Cl_Δ. The heuristics given in [5] state that the probability that the p-rank is equal to r is

$$\mathrm{prob}(p\text{-rank of } Cl_\Delta = r) = \frac{\eta_\infty(p)}{p^{r^2}\eta_r(p)^2} \ . \tag{0.4}$$

We define $p_{l,r}(x)$ as the observed ratio of discriminants less than x with l-rank equal to r divided by the conjectured probability shown in (0.4). As x increases, we would expect the value of $p_{l,r}(x)$ to approach 1.

In Table 5, we present values of $p_{l,r}(x)$ for various values of small primes l and $r = 2, 3, 4$. As expected, the values tend to approach 1 from below fairly smoothly, but slowly.

First Occurrences of Non-cyclic p-Sylow Subgroups

In [3], Buell looked at what he called "exotic" groups, particular non-cyclic p-Sylow subgroups for various odd primes p. Following Buell, we have recorded both the first occurrence and the total number of discriminants for which a specific p-Sylow subgroup is non-cyclic. When dealing with the prime $p = 2$, we consider only the 2-Sylow subgroup of the principal genus (the subgroup of squares) of the class group, as was done in [6] and [3].

In Tables 6 and 7, we present the discriminants Δ with the smallest absolute value for which Cl_Δ has a rank 2 p-Sylow subgroup of the form $C(p^{e_1}) \times C(p^{e_2})$. Table 6 lists data for $p = 2$, and Table 7 lists data for odd primes p. We have tabulated and displayed those discriminants where $\Delta \equiv 0 \pmod 4$ and those where $\Delta \equiv 1 \pmod 4$ separately. We also list the number of discriminants $|\Delta| < 10^{11}$ for which each p-Sylow subgroup has the specified structure. We found several fields for which the p-Sylow subgroup has rank 2 for all odd primes $p \leq 389$.

In Tables 8 and 9, we present the discriminants Δ with the smallest absolute value for which Cl_Δ has a rank 3 p-Sylow subgroup of the form $C(p^{e_1}) \times C(p^{e_2}) \times C(p^{e_3})$. Table 8 lists data for $p = 2$, and Table 9 lists data for odd primes p. Once again, we list discriminants in different congruence classes separately, and also the number of discriminants for which each p-Sylow subgroup has the specified structure. We found fields with p-Sylow subgroups of rank 3 for all odd primes $p \leq 13$. Although fields with 11 and 13-Sylow subgroups of rank 3 were already known [8, 9], the discriminants we found are unconditionally the smallest in absolute value of any fields with these properties.

In Table 10, we present discriminants Δ with the smallest absolute value for which $Cl - \Delta$ has a rank 4 2-Sylow subgroup. We found numerous examples of fields with rank 4 3-Sylow subgroups, listed in Table 11. We did not observe any fields with p-rank equal to 4 for $p > 3$. In Table 10, we list similar data for $p = 2$.

In Table 12 we present the first occurrences of doubly non-cyclic class groups, and in Table 13 we present the first occurrences of trebly non-cyclic class groups. The most "exotic" of these class groups, for $\Delta = -61164913211$, is isomorphic to $C(3 \cdot 7 \cdot 19) \times C(3 \cdot 7 \cdot 19)$. In addition, we were able to find 4 discriminants for which the corresponding class groups are quadruply non-cyclic with respect to the primes $2, 3, 5$ and 7. The smallest of these discriminants is $\Delta = -20777253551$ with $Cl_\Delta \cong C(4 \cdot 3 \cdot 5 \cdot 7) \times C(4 \cdot 3 \cdot 5 \cdot 7)$.

Number of Generators

In [1], Bach proved a theorem stating that under the ERH, prime ideals of norm less than $6 \log^2 |\Delta|$ are sufficient to generate the class group of a quadratic field. However, in [4], a tighter bound of $O(\log^{1+\epsilon} |\Delta|)$ was conjectured. Other authors, such as [3] and [6], have observed that in practice, Bach's bound seems to be excessive and attempt to find a constant c for which the tighter bound could hold.

We define $\max_p(\Delta)$ as the maximum norm of the prime ideals required to generate the class group of $\mathbb{Q}(\sqrt{\Delta})$. If Bach's theorem is true, we would expect that $\max_p(\Delta)/\log^2|\Delta| \leq 6$. To test this theorem, we maintained values of $\max_p(\Delta)$ for all discriminants Δ with $0 < |\Delta| < 10^{11}$. In order to test the tighter bound given in [4], we tried to find the magnitude of the constant c for which $\max_p(\Delta) \leq c \log|\Delta|$. To do this, we looked at the ratio of $\max_p(\Delta)/\log|\Delta|$.

Throughout our computations, the maximum value of $\max_p(\Delta)$ we found was 353 for $\Delta = -42930759883$ and $\Delta = -88460711448$. The maximum value of $\max_p(\Delta)/\log^2|\Delta|$ was 0.780042... for the discriminant $\Delta = -424708$, and the average value was 0.02481.... The maximum value of $\max_p(\Delta)/\log|\Delta|$ was 14.41825... for the discriminant $\Delta = -42930759883$, and the average value was 0.60191..... The maximum value of $\max_p(\Delta)/\log^2|\Delta|$ remained constant for most of the computation, whereas the maximum of $\max_p(\Delta)/\log|\Delta|$ increased very slowly, suggesting that a bound of $O(\log^{1+\epsilon}|\Delta|)$ may indeed be the truth. Table 16 lists the complete data for $\max_p(\Delta)$ and both ratios.

Following Buell [3], we also kept track of the first occurrences and total number of discriminants for which all prime ideals of norm up to a certain bound were necessary, with the maximum norm found being 353. Table 14 lists these values. We found that the total number of discriminants requiring all prime ideals of norm up to a prime p tended to decrease as p increased.

We also looked at the number of prime ideals that were required to generate the class group, listed in Table 15. The maximum number of prime ideals required to generate all discriminants Δ for $0 < |\Delta| < 10^{11}$ was 25 for the discriminant $\Delta = -75948116920$, but on average only 3.31359... were required.

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A Appendix

Table 1. Successive $L(1, \chi)$ maxima, $\varDelta \equiv 1 \pmod 8$

$\varDelta$	$L(1, \chi)$	ULI	$L_D(1)$	ULI_D
1514970551	7.49759	0.68985	11.24638	0.67578
2438526191	7.52739	0.68757	11.29108	0.67394
2570169839	7.56669	0.69062	11.35004	0.67697
2772244991	7.58892	0.69186	11.38339	0.67825
3555265271	7.59038	0.68945	11.38556	0.67607
5111994359	7.64749	0.69097	11.47123	0.67784
6194583071	7.69307	0.69318	11.53961	0.68016
7462642151	7.70257	0.69221	11.55386	0.67934
7979490791	7.70933	0.69217	11.56400	0.67934
8462822759	7.77325	0.69733	11.65988	0.68445
12123145319	7.80183	0.69642	11.70274	0.68381
13005495359	7.82594	0.69790	11.73890	0.68531
17833071959	7.89105	0.70071	11.83658	0.68829
29414861999	7.89941	0.69683	11.84911	0.68480
35535649679	7.94608	0.69923	11.91912	0.68728
42775233959	7.99504	0.70187	11.99257	0.68998
45716419031	8.09414	0.70996	12.14121	0.69798

Table 2. Successive $L(1, \chi)$ minima, $\varDelta \equiv 5 \pmod 8$

$\varDelta$	$L(1, \chi)$	LLI	$L_D(1)$	LLI_D
1930143763	0.18764	1.24439	0.28146	1.90488
2426489587	0.18655	1.24146	0.27982	1.89987
2562211723	0.18470	1.23020	0.27706	1.88252
3030266803	0.18445	1.23160	0.27668	1.88429
3416131987	0.18152	1.21415	0.27227	1.85734
6465681643	0.18082	1.22069	0.27122	1.86602
6623767483	0.17973	1.21375	0.26959	1.85536
15442196323	0.17843	1.21922	0.26765	1.86211
21538327507	0.17609	1.20857	0.26413	1.84525
45640185427	0.17604	1.22007	0.26406	1.86152
84291143203	0.17599	1.22914	0.26398	1.87437
85702502803	0.17448	1.21885	0.26172	1.85865

Table 3. Values of $p_l(x)$

x	$p_3(x)$	$p_5(x)$	$p_7(x)$	$p_{11}(x)$	$p_{13}(x)$	$p_{17}(x)$	$p_{19}(x)$
1000000000	0.97327	0.99348	0.99609	0.99576	0.99489	0.99474	0.99347
2000000000	0.97624	0.99453	0.99687	0.99664	0.99621	0.99585	0.99522
3000000000	0.97783	0.99515	0.99737	0.99701	0.99666	0.99646	0.99601
4000000000	0.97888	0.99558	0.99760	0.99724	0.99708	0.99672	0.99657
5000000000	0.97966	0.99586	0.99778	0.99751	0.99738	0.99698	0.99698
6000000000	0.98029	0.99610	0.99786	0.99769	0.99757	0.99724	0.99718
7000000000	0.98080	0.99622	0.99791	0.99780	0.99771	0.99742	0.99737
8000000000	0.98122	0.99635	0.99800	0.99795	0.99787	0.99753	0.99745
9000000000	0.98159	0.99645	0.99808	0.99803	0.99799	0.99763	0.99757
10000000000	0.98191	0.99653	0.99818	0.99810	0.99812	0.99771	0.99770
20000000000	0.98391	0.99712	0.99861	0.99852	0.99853	0.99823	0.99824
30000000000	0.98496	0.99744	0.99876	0.99875	0.99876	0.99850	0.99852
40000000000	0.98567	0.99761	0.99887	0.99890	0.99889	0.99871	0.99868
50000000000	0.98619	0.99776	0.99896	0.99901	0.99900	0.99884	0.99880
60000000000	0.98661	0.99786	0.99901	0.99904	0.99906	0.99891	0.99888
70000000000	0.98695	0.99796	0.99905	0.99908	0.99912	0.99902	0.99893
80000000000	0.98723	0.99804	0.99909	0.99912	0.99916	0.99907	0.99902
90000000000	0.98748	0.99810	0.99913	0.99917	0.99920	0.99911	0.99906
100000000000	0.98770	0.99815	0.99915	0.99919	0.99924	0.99914	0.99910

Table 4. Number of noncyclic odd parts of class groups

x	total	non-cyclic	percent	c(x)
1000000000	303963510	5585092	1.83742	1.00414
2000000000	607927095	11356654	1.86809	1.00383
3000000000	911890759	17182389	1.88426	1.00366
4000000000	1215854223	23041817	1.89511	1.00355
5000000000	1519817699	28923395	1.90308	1.00347
6000000000	1823781240	34822620	1.90936	1.00341
7000000000	2127745010	40736296	1.91453	1.00336
8000000000	2431708386	46659753	1.91881	1.00331
9000000000	2735672001	52600902	1.92278	1.00327
10000000000	3039635443	58544601	1.92604	1.00324
20000000000	6079271092	118313612	1.94618	1.00303
30000000000	9118906425	178447518	1.95690	1.00292
40000000000	12158541989	238793386	1.96400	1.00285
50000000000	15198177465	299290965	1.96926	1.00280
60000000000	18237813070	359892824	1.97333	1.00275
70000000000	21277448334	420584966	1.97667	1.00272
80000000000	24317083860	481364092	1.97953	1.00269
90000000000	27356719791	542201863	1.98197	1.00267
100000000000	30396355052	603101904	1.98413	1.00264

Table 5. Values of $p_{l,r}(x)$

x	$pr_{3,2}(x)$	$pr_{5,2}(x)$	$pr_{7,2}(x)$	$pr_{11,2}(x)$	$pr_{13,2}(x)$	$pr_{3,3}(x)$	$pr_{5,3}(x)$	$pr_{7,3}(x)$	$pr_{3,4}(x)$
1000000000	0.84360	0.95708	0.96014	0.94248	0.92552	0.44305	0.92707	0.33360	0.08031
2000000000	0.86065	0.96351	0.96570	0.95530	0.94096	0.49026	0.90254	0.38920	0.08031
3000000000	0.86959	0.96705	0.97262	0.96803	0.94671	0.51636	0.89927	0.59306	0.10708
4000000000	0.87560	0.96977	0.97550	0.97268	0.95343	0.53591	0.91726	0.63940	0.12047
5000000000	0.88013	0.97125	0.97670	0.97199	0.95713	0.55085	0.93099	0.73392	0.11244
6000000000	0.88365	0.97289	0.97885	0.97394	0.95893	0.56132	0.93770	0.83400	0.13385
7000000000	0.88658	0.97382	0.98039	0.97483	0.96086	0.57142	0.93968	0.82605	0.16062
8000000000	0.88904	0.97467	0.98048	0.97631	0.96206	0.58047	0.93627	0.83400	0.19074
9000000000	0.89126	0.97566	0.98150	0.97711	0.96586	0.58722	0.93415	0.79075	0.20524
10000000000	0.89309	0.97642	0.98224	0.97917	0.96762	0.59382	0.93394	0.73392	0.20881
20000000000	0.90470	0.98042	0.98737	0.98407	0.97719	0.63193	0.93614	0.76172	0.24896
30000000000	0.91096	0.98264	0.98868	0.98628	0.98298	0.65288	0.94064	0.78581	0.24361
40000000000	0.91516	0.98384	0.98918	0.98709	0.98517	0.66798	0.95491	0.83956	0.25298
50000000000	0.91828	0.98481	0.98996	0.98755	0.98684	0.67905	0.95385	0.84956	0.26503
60000000000	0.92072	0.98541	0.99060	0.98830	0.98749	0.68821	0.95707	0.87106	0.27707
70000000000	0.92272	0.98605	0.99119	0.98845	0.98738	0.69563	0.96477	0.87530	0.29141
80000000000	0.92444	0.98653	0.99168	0.98917	0.98795	0.70175	0.96398	0.89098	0.29916
90000000000	0.92591	0.98704	0.99198	0.98966	0.98824	0.70727	0.96729	0.91060	0.31143
100000000000	0.92721	0.98743	0.99223	0.99025	0.98900	0.71201	0.96636	0.91628	0.31803

Table 6. Non-cyclic rank 2 2-Sylow subgroups

e_1	e_2	first even Δ	# even Δ	first odd Δ	# odd Δ
1	1	6052	224858692	2379	365605352
2	1	6392	169585266	5795	275891156
2	2	25988	13872240	32331	22597752
3	1	7544	84818703	4895	137916207
3	2	118040	10406948	22127	16935890
3	3	636664	868738	618947	1411876
4	1	39236	42395895	10295	68919103
4	2	264452	5205067	103727	8468289
4	3	1353316	650587	804639	1059545
4	4	4126328	54346	2365599	87923
5	1	145604	21199250	60695	34479271
5	2	605816	2601468	310295	4235704
5	3	3118916	325905	1008095	528790
5	4	14060036	40732	13263095	66204
5	5	53231864	3441	22128095	5589
6	1	312584	10599473	187239	17238175

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Table 6 – continued from previous page

e_1	e_2	first even Δ	# even Δ	first odd Δ	# odd Δ
6	2	2472824	1299916	968495	2116240
6	3	8049284	162125	3194495	263789
6	4	49563236	20443	12993671	32944
6	5	115629944	2535	122764631	4238
6	6	979202552	197	1059634567	346
7	1	1297544	5299113	535871	8621884
7	2	5407736	649014	3177095	1058767
7	3	43637624	81733	6342959	131784
7	4	217291076	10244	24475919	16696
7	5	585612296	1230	353879327	2053
7	6	3777569528	129	1331102631	243
7	7	49964393960	9	4487508695	12
8	1	4765316	2647387	2009111	4310798
8	2	23989796	325459	6851831	528795
8	3	112475684	40794	27310895	66516
8	4	415337096	4998	155791391	8100
8	5	1872668936	550	700226279	993
8	6	19379520584	51	2500463471	95
8	7	43627697252	3	20564153183	7
8	8	28148188439	2	*	*
9	1	16899704	1323915	5266439	2155276
9	2	106946936	162295	27704351	265054
9	3	348733304	19879	153404279	32869
9	4	1525528196	2108	434237639	3892
9	5	8793326372	171	1697415695	357
9	6	40109627876	8	9658267583	25
9	7	*	*	36156111551	3
10	1	74198264	663388	22858871	1074161
10	2	200896484	80167	105684095	131204
10	3	1186927304	8590	342897959	15782
10	4	5778168824	676	1544290079	1556
10	5	27799085816	26	10843705871	123
10	6	*	*	32772714719	7
11	1	236054264	325960	75612599	534683
11	2	827711876	34523	304561631	63178
11	3	3722450696	2702	1324096199	6197
11	4	14034192644	117	4543687511	478
11	5	92221912184	1	29925386231	21
12	1	929170436	140086	280073351	256954
12	2	3562207736	11149	1120758911	24749
12	3	17874504584	570	5855914895	1867
12	4	82876399304	2	20975257511	101
12	5	*	*	90998785895	1
13	1	3989574536	46283	966467519	101294
13	2	15271434884	1968	4429883519	7587

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Table 6 – continued from previous page

e_1	e_2	first even Δ	# even Δ	first odd Δ	# odd Δ
13	3	73338178436	23	15406567679	416
13	4	*	*	94172111879	3
14	1	13471444964	8346	3899095199	30950
14	2	53549964536	96	12633802271	1477
14	3	*	*	70219409399	9
15	1	49186240484	344	15649176791	6194
15	2	*	*	63808583879	40
16	1	*	*	54229304951	160

Table 7. Non-cyclic rank 2 p-Sylow subgroups

p	e_1	e_2	first even Δ	# even Δ	first odd Δ	# odd Δ
3	1	1	3896	109145016	4027	219959274
3	2	1	27656	48500803	3299	97662978
3	2	2	208084	1346135	134059	2707009
3	3	1	55316	16171603	17399	32557316
3	3	2	998708	598440	351751	1200372
3	3	3	39337384	16586	6207263	33370
3	4	1	462356	5390426	29399	10845649
3	4	2	4279448	200522	1332167	400590
3	4	3	88848836	7344	41361815	14856
3	4	4	1271559208	170	136071631	381
3	5	1	3935384	1798630	508847	3616620
3	5	2	36356456	65603	15042011	133749
3	5	3	576873236	2173	152637311	4676
3	5	4	16887409796	45	4301015239	115
3	5	5	*	*	6743415071	4
3	6	1	28026164	594799	3582743	1203905
3	6	2	263591156	19822	19180391	42755
3	6	3	2671485416	443	636617543	1279
3	6	4	49547047976	1	7274282423	32
3	7	1	232838744	177273	32681951	386998
3	7	2	2175729716	3868	167885231	11647
3	7	3	17668343384	42	3541241903	269
3	7	4	*	*	47649110911	3
3	8	1	1723181864	35071	98311919	106366
3	8	2	17082145064	241	1173834359	2289
3	8	3	*	*	37703425007	18
3	9	1	11132690456	2153	1106108639	20187
3	9	2	93287426216	1	11901791639	221
3	9	3	*	*	60543925679	1
3	10	1	98284577816	2	8795475911	2039
3	10	2	*	*	65798421911	4

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Table 7 – continued from previous page

p	e_1	e_2	first even Δ	# even Δ	first odd Δ	# odd Δ
3	11	1	*	*	52623967679	21
5	1	1	17944	15856558	11199	31744688
5	2	1	178004	3803543	50783	7615998
5	2	2	9623444	25190	1390367	50668
5	3	1	2189204	759590	621599	1522101
5	3	2	273928024	5745	52456111	12042
5	3	3	13603495364	16	1068156239	70
5	4	1	56245556	142709	5820119	297755
5	4	2	2194276244	651	290810159	1841
5	4	3	*	*	10036313687	8
5	5	1	1163891636	15455	88527911	46929
5	5	2	26611903016	9	5180829911	129
5	6	1	25411429364	205	1614153239	3578
5	6	2	*	*	75913193999	1
5	7	1	*	*	48662190359	51
7	1	1	159592	4184728	63499	8362139
7	2	1	890984	682702	480059	1364713
7	2	2	288854504	1668	59288543	3485
7	3	1	50642024	91897	4603007	190857
7	3	2	5468598824	115	528784319	397
7	3	3	*	*	40111506371	1
7	4	1	1157188724	6268	172820591	19925
7	4	2	75003362216	1	16336216607	14
7	5	1	64461971636	18	5800676279	672
11	1	1	580424	689534	65591	1375345
11	2	1	24557096	66089	7948999	135668
11	2	2	8124316712	19	4536377039	69
11	3	1	712328756	2815	218130623	8793
11	3	2	*	*	91355041631	1
11	4	1	89983172564	1	7219509359	95
13	1	1	703636	353317	228679	706372
13	2	1	86189912	27111	14127343	56801
13	2	2	15290030216	2	10692322055	12
13	3	1	5247449576	493	781846103	2375
13	4	1	*	*	55385334839	10
17	1	1	4034356	121125	1997799	241139
17	2	1	558578648	5817	43780679	13620
17	2	2	*	*	94733724779	1
17	3	1	28205334296	10	5767994839	201
19	1	1	3419828	77093	373391	154983
19	2	1	921151124	2921	17803439	7275
19	3	1	*	*	5862529559	69
23	1	1	11137012	35742	7472983	71933
23	2	1	1188873236	783	510491431	2348
23	3	1	*	*	74447537447	2

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Table 7 – continued from previous page

p	e_1	e_2	first even Δ	# even Δ	first odd Δ	# odd Δ
29	1	1	16706324	13833	20113607	28162
29	2	1	5614832984	137	296873471	534
31	1	1	96468488	10610	11597903	21646
31	2	1	14560212776	62	362103671	367
37	1	1	25162772	5105	51461727	10719
37	2	1	33184320308	6	2793641999	99
41	1	1	99141272	3357	6112511	6871
41	2	1	29030848244	6	12558317543	49
43	1	1	66614312	2678	39405967	5631
43	2	1	*	*	28602441479	26
47	1	1	1054312388	1800	57403799	3949
47	2	1	65816894324	2	20751947191	18
53	1	1	777263864	1097	26675327	2367
53	2	1	*	*	34862413351	3
59	1	1	244858136	659	133943879	1532
59	2	1	*	*	65887828631	2
61	1	1	1264482536	556	137323663	1383
67	1	1	1516640872	381	448220959	834
71	1	1	839514836	286	198786779	684
73	1	1	420255476	275	483264167	622
79	1	1	5114393428	154	888934163	445
83	1	1	2390420804	136	884989055	354
89	1	1	2339707096	99	1941485183	259
97	1	1	9388308724	70	2179032511	177
101	1	1	4293806984	45	758562611	164
103	1	1	19084053944	45	787024943	132
107	1	1	4576627816	40	4041299887	125
109	1	1	2202664232	30	4903396807	97
113	1	1	3422486836	30	1047199379	71
127	1	1	7127111912	21	3482629127	46
131	1	1	16018714472	12	2884161823	45
137	1	1	37914915092	4	4549823483	38
139	1	1	50553654520	4	8396560295	29
149	1	1	56336668888	4	15233330011	20
151	1	1	42941394424	4	13310472899	19
157	1	1	19416052676	5	15661511531	24
163	1	1	46586000024	3	10302820679	15
167	1	1	20926233044	2	22669688623	13
173	1	1	84419230376	4	14602373903	14
179	1	1	89298106628	2	28362963611	9
181	1	1	89809227124	1	8991716639	15
191	1	1	*	*	48122759783	3
193	1	1	28354858472	1	84647431783	2
197	1	1	*	*	66490566011	3
199	1	1	*	*	6561724871	5

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Table 7 – continued from previous page

p	e_1	e_2	first even Δ	# even Δ	first odd Δ	# odd Δ
211	1	1	97297226504	1	18146008687	1
223	1	1	*	*	36799898071	2
229	1	1	*	*	89550601631	1
233	1	1	*	*	29922371399	4
239	1	1	84169153736	1	55757811107	1
241	1	1	*	*	74882513855	1
251	1	1	*	*	78181110431	1
257	1	1	*	*	23738884679	1
263	1	1	*	*	37893813311	3
269	1	1	*	*	11129396567	1
283	1	1	*	*	94175615183	1
349	1	1	*	*	32819826815	1
389	1	1	*	*	85401404639	1

Table 8. Non-cyclic rank 3 2-Sylow subgroups

e_1	e_2	e_3	first even Δ	# even Δ	first odd Δ	# odd Δ
1	1	1	1148984	3726047	1295823	6305389
2	1	1	568888	3289034	503659	5549219
2	2	1	3040888	404125	2209467	686914
2	2	2	29418872	6050	31078723	10914
3	1	1	550712	1644999	1696071	2773638
3	2	1	5235592	302160	1456131	514771
3	2	2	58984568	5267	38432395	9590
3	3	1	42747512	25103	15254499	42479
3	3	2	180245764	647	306703595	1202
3	3	3	26037089032	9	3072761723	28
4	1	1	2256376	821518	863455	1384326
4	2	1	4605572	150924	4312495	256972
4	2	2	84855928	2696	113368287	4709
4	3	1	32985032	18835	12315783	32007
4	3	2	709356440	514	262912611	922
4	3	3	12536161012	8	968674895	24
4	4	1	194468984	1558	63983699	2746
4	4	2	599039224	41	1348092695	79
4	4	3	48635750392	3	15453558059	2
5	1	1	3600632	410300	2600247	692368
5	2	1	6030584	75956	11888359	128596
5	2	2	656353752	1305	176472095	2411
5	3	1	110604856	9485	89794955	15987
5	3	2	942374776	250	530051079	440
5	3	3	36300169992	3	29812383539	7
5	4	1	306870392	1118	52959695	2000

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Table 8 – continued from previous page

e_1	e_2	e_3	first even Δ	# even Δ	first odd Δ	# odd Δ
5	4	2	1950599672	24	745029527	55
5	4	3	19285756744	4	26840480251	2
5	5	1	632499896	94	3153932195	174
5	5	2	20634196984	1	30024717407	5
6	1	1	9601544	205615	3285399	347346
6	2	1	40329464	38284	12870695	63968
6	2	2	475618808	681	116917743	1251
6	3	1	110476484	4736	80285439	7937
6	3	2	1685435012	138	245004591	245
6	3	3	19231492484	6	5154637111	6
6	4	1	1028005256	577	709215951	1004
6	4	2	18928358948	11	1745876431	32
6	5	1	11230043192	60	2025881495	125
6	5	2	*	*	12135120919	7
6	6	1	41596381252	4	13864237495	6
7	1	1	29971256	102647	11621255	173667
7	2	1	114856964	18976	39546239	32072
7	2	2	1598656804	327	1153171407	660
7	3	1	518626616	2377	104663495	4055
7	3	2	3452129864	50	3937488031	116
7	3	3	17129098616	1	43500603367	3
7	4	1	1985708324	312	805283687	510
7	4	2	30957938552	6	37875548711	10
7	5	1	20184931576	28	2561173655	52
7	5	2	*	*	71448310047	1
7	6	1	71493317624	2	11605902591	7
7	7	1	*	*	42732217895	1
8	1	1	65195576	51426	41667695	86614
8	2	1	322681316	9430	162267599	16034
8	2	2	1927781624	155	1195516295	299
8	3	1	1264546436	1139	698196239	2114
8	3	2	18368038136	16	7762724495	51
8	3	3	40374695204	1	19206387503	2
8	4	1	6533780984	136	3423173871	228
8	4	2	97911313784	2	36780288287	3
8	5	1	21419706104	10	8359014839	25
8	6	1	*	*	79607470655	1
9	1	1	317455544	25910	118575119	43309
9	2	1	1697181944	4609	417704759	8269
9	2	2	15725508164	59	9044868391	125
9	3	1	5874224996	493	2678195351	826
9	3	2	38767885124	6	9656429351	19
9	4	1	35512754756	26	8510136095	59
9	4	2	91670247044	1	*	*
9	5	1	*	*	20210353631	3

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Table 8 – continued from previous page

e_1	e_2	e_3	first even Δ	# even Δ	first odd Δ	# odd Δ
10	1	1	1387470776	12613	408615119	22010
10	2	1	4688751044	1866	2482130199	3424
10	2	2	25496650616	12	12194979695	42
10	3	1	20948614904	111	5655701831	316
10	3	2	*	*	38525569031	5
10	4	1	89449248164	2	25800653711	20
11	1	1	4022802824	5460	1345931495	9316
11	2	1	19408564964	471	7545348695	1236
11	2	2	*	*	27487687295	13
11	3	1	59346834104	5	25662174551	80
12	1	1	16063397444	1308	5744234831	3128
12	2	1	65603208824	25	16360877231	281
12	3	1	*	*	91844298959	1
13	1	1	57410101124	65	22371341879	763
13	2	1	*	*	59760022871	5
14	1	1	*	*	58698960239	18

Table 9. Non-cyclic rank 3 p-Sylow subgroups

p	e_1	e_2	e_3	first even Δ	# even Δ	first odd Δ	# odd Δ
3	1	1	1	4447704	352660	4897363	728836
3	2	1	1	5288968	169861	3321607	349337
3	2	2	1	145519608	6341	101375499	12944
3	2	2	2	3457439416	18	364435991	35
3	3	1	1	12755172	56315	5153431	117132
3	3	2	1	57236692	2828	79378899	5746
3	3	2	2	18741973496	9	11037391871	8
3	3	3	1	1940867992	74	559587163	141
3	3	3	2	*	*	20687610651	1
3	4	1	1	35180884	18709	13275687	38812
3	4	2	1	192757064	922	53209523	1885
3	4	2	2	12251300788	4	9766538987	7
3	4	3	1	2245873412	29	522302531	67
3	4	4	1	*	*	26320580987	1
3	5	1	1	111442868	6368	32852423	13173
3	5	2	1	3130903236	272	413771887	625
3	5	2	2	*	*	45248632247	2
3	5	3	1	43721231572	5	2232519167	15
3	6	1	1	509049176	1935	124438679	4084
3	6	2	1	19996254456	61	376424303	165
3	6	2	2	*	*	9483757583	1
3	6	3	1	27291040424	1	53192765699	3
3	7	1	1	6382094504	373	461309711	1183

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Table 9 – continued from previous page

p	e_1	e_2	e_3	first even Δ	# even Δ	first odd Δ	# odd Δ
3	7	2	1	33828950744	4	4163792239	35
3	8	1	1	20594835764	24	5347129751	255
3	8	2	1	*	*	59714529551	3
3	9	1	1	*	*	12792023879	22
5	1	1	1	11203620	4935	18397407	10078
5	2	1	1	272394484	1245	51213139	2556
5	2	2	1	7095550408	9	6896149079	14
5	3	1	1	300240404	243	145367147	506
5	3	2	1	49468612564	1	29867315295	2
5	4	1	1	5871738932	32	3511272455	75
5	5	1	1	*	*	25384593659	5
7	1	1	1	1898879592	222	501510767	485
7	2	1	1	2760876184	34	648153647	70
7	3	1	1	32727392168	4	19379510159	9
11	1	1	1	3035884424	6	23235125867	5
13	1	1	1	*	*	38630907167	2

Table 10. Non-cyclic rank 4 2-Sylow subgroups

e_1	e_2	e_3	e_4	first even Δ	# even Δ	first odd Δ	# odd Δ
1	1	1	1	471960184	7712	197731195	13215
2	1	1	1	498994552	7298	511858407	12419
2	2	1	1	942647416	1084	914157695	1877
2	2	2	1	4647849848	23	4924087483	41
3	1	1	1	194401336	3535	349519339	6249
3	2	1	1	1649835652	784	872841047	1390
3	2	2	1	11726216888	26	2659965695	37
3	3	1	1	7736922872	63	4199412607	120
3	3	2	1	52035316984	4	17184321295	6
4	1	1	1	934558264	1737	141244707	3157
4	2	1	1	437474872	387	440663695	634
4	2	2	1	5648488952	17	1992071683	21
4	3	1	1	7171718852	44	530870223	80
4	3	2	1	68445834520	1	18394768247	4
4	3	3	1	*	*	71657045499	1
4	4	1	1	70040577764	3	1737119891	3
4	4	2	1	*	*	35723521195	2
5	1	1	1	1110786104	870	385521331	1561
5	2	1	1	4943558040	186	2034087987	339
5	2	2	1	33861890488	5	35738073111	8
5	3	1	1	16381065848	18	2547515451	48
5	3	2	1	35219141368	3	41645702735	3
5	4	1	1	18891451144	4	26517688015	2

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Table 10 – continued from previous page

e_1	e_2	e_3	e_4	first even Δ	# even Δ	first odd Δ	# odd Δ
6	1	1	1	1036693796	426	905764295	768
6	2	1	1	4350311096	102	4942746003	181
6	2	2	1	43143969656	4	8628191135	5
6	3	1	1	24951539576	13	29583016707	16
6	3	2	1	*	*	53490173795	1
6	4	1	1	76332750328	1	68390991723	2
6	5	1	1	31903643768	1	42458106639	1
7	1	1	1	726515384	217	816714055	401
7	2	1	1	4610055416	46	672821903	77
7	2	2	1	40338576376	2	75905537331	3
7	3	1	1	19570359032	9	16839000895	14
7	4	1	1	75305261828	2	*	*
8	1	1	1	12243038648	110	2436431439	183
8	2	1	1	12815163704	32	8573244695	46
8	2	2	1	92790235832	1	91630708055	3
8	3	1	1	61265888312	2	51413000223	2
8	4	1	1	95938795256	1	*	*
9	1	1	1	14577769412	57	16567132647	103
9	2	1	1	44331931256	7	15026935655	12
9	3	1	1	*	*	47502531911	1
10	1	1	1	34219906616	16	13189888895	40
10	2	1	1	*	*	73131029751	1
11	1	1	1	83893756964	2	31482746399	12

Table 11. Non-cyclic rank 4 p-Sylow subgroups

p	e_1	e_2	e_3	e_4	first even Δ	# even Δ	first odd Δ	# odd Δ
3	1	1	1	1	2520963512	62	653329427	172
3	2	1	1	1	11451958228	27	3972542271	83
3	2	2	1	1	64435895236	1	32543535351	3
3	3	1	1	1	26041127732	8	5288116947	21
3	3	2	1	1	34245189208	1	*	*
3	4	1	1	1	3146813128	6	35684560479	5
3	5	1	1	1	17346090376	2	7993105123	4
3	6	1	1	1	*	*	76951070303	1

Table 12. Doubly non-cyclic class groups

p_1	p_2	first even Δ	# even Δ	first odd Δ	# odd Δ
2	3	64952	11083237	104255	18064971

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Table 12 – continued from previous page

p_1	p_2	first even Δ	# even Δ	first odd Δ	# odd Δ
2	5	472196	1218252	280847	1977804
2	7	858296	291590	465719	474962
2	11	11221832	43871	10724727	71453
2	13	34388612	21718	13725759	35974
2	17	62975684	7229	50684339	11822
2	19	287484728	4387	41698223	7444
2	23	427072292	1960	206527919	3432
2	29	1189666616	723	621746551	1171
2	31	1893349604	505	410359895	867
2	37	3833690756	219	1839127055	440
2	41	5503733780	146	2620148255	255
2	43	6036095204	79	3155959391	231
2	47	5201404616	70	4812997295	149
2	53	17170629304	29	2436482551	72
2	59	12050201444	22	9075455255	37
2	61	19120092536	19	19914584807	23
2	67	22395736184	7	2962630655	23
2	71	15544112516	3	17401096571	11
2	73	18251641796	2	34627398895	17
2	79	63115620356	1	33743280671	10
2	83	52717441208	2	44282533439	6
2	89	57218613128	3	24120821559	7
2	97	*	*	55968627055	4
2	101	*	*	69000110911	1
3	5	2766392	350787	119191	718055
3	7	16053944	84140	3561799	172141
3	11	22297448	12343	14898623	26409
3	13	70887272	6194	39186347	12982
3	17	142736408	1969	188315447	4226
3	19	372243764	1261	234113631	2626
3	23	1675854452	534	351756527	1230
3	29	3395393108	167	557577743	446
3	31	3792995864	122	386659943	314
3	37	6112785556	55	1455428855	146
3	41	17658330596	26	1166119039	80
3	43	3286197848	30	4075192859	71
3	47	16964359736	14	485163311	44
3	53	41696300984	5	457096511	23
3	59	49943038232	5	10227491279	19
3	61	32515774996	4	8522929927	17
3	67	84253538216	2	26792580191	8
3	71	*	*	17614533947	8
3	79	85480238756	1	51762875627	6
3	83	*	*	50476998239	4
3	97	*	*	43344787079	2

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Table 12 – continued from previous page

p_1	p_2	first even Δ	# even Δ	first odd Δ	# odd Δ
3	71	*	*	17614533947	8
3	73	*	*	11752995103	9
3	79	85480238756	1	51762875627	6
3	83	*	*	50476998239	4
3	97	*	*	43344787079	2
3	103	*	*	93069031703	1
3	109	*	*	35029686023	1
3	113	*	*	56428950647	1
5	7	45324248	8962	19399067	18665
5	11	195367988	1241	112179371	2814
5	13	644440376	567	94672727	1348
5	17	714706004	171	135145159	422
5	19	3925533652	102	965381231	271
5	23	14260068616	33	336603767	108
5	29	15541379720	12	10138338695	29
5	31	11788579624	8	17205833747	23
5	37	10719968216	3	16249120831	8
5	41	*	*	26948199679	8
5	43	51986729896	1	71114945339	1
5	47	*	*	8182208159	4
5	53	*	*	22759605719	2
5	71	*	*	14917874303	1
5	73	*	*	63515115611	1
7	11	629704808	247	235436591	584
7	13	1419402728	102	1580010631	281
7	17	6198957812	37	1851928807	87
7	19	24082268968	7	5166049215	53
7	23	22198579640	10	2591136407	20
7	29	*	*	21164450935	8
7	31	18704562356	1	68200813691	1
7	37	*	*	49918973471	1
7	43	*	*	57006644887	1
7	47	*	*	98533572251	1
11	13	31664474564	11	13609279311	31
11	17	50159859416	2	41219120419	10
11	19	*	*	19439678123	2
11	23	*	*	94266055451	1
13	17	72831993636	1	41507696303	4
13	19	*	*	75779342435	2
17	23	*	*	54134972891	1

Table 13. Trebly non-cyclic class groups

p_1	p_2	p_3	first even Δ	# even Δ	first odd Δ	# odd Δ
2	3	5	37892516	21184	23677127	35378
2	3	7	108485576	4864	82966895	8337
2	3	11	1872785336	581	264640055	1135
2	3	13	7213488644	266	1047903271	512
2	3	17	4928392004	79	1882856559	163
2	3	19	17805303416	29	6136370399	104
2	3	23	23420913668	11	15603393095	32
2	3	29	80314582984	1	15426649911	9
2	3	31	66961530788	1	22141447799	3
2	3	37	*	*	83842784855	1
2	3	43	*	*	88786054727	1
2	3	53	*	*	96869926295	1
2	5	7	1282481528	379	2003704023	745
2	5	11	13208419364	35	7602070071	74
2	5	13	20738568548	9	17041306931	30
2	5	17	*	*	61842180179	3
2	5	19	*	*	75528143095	1
2	7	11	18638540036	7	19274727711	13
2	7	13	32415492836	2	16664272895	4
2	7	17	*	*	61444342919	1
2	7	19	*	*	67611298199	1
3	5	7	6890424056	78	1475373743	264
3	5	11	49957566964	6	4643885759	30
3	5	13	84831842696	2	13308756863	14
3	5	17	*	*	60235736039	5
3	7	11	42843308072	2	*	*
3	7	13	*	*	38986878143	4
3	7	19	*	*	61164913211	1

Table 14. First Δ needing prime ideals of norm up to p

p	first even Δ	# even Δ	first odd Δ	# odd Δ
3	4	589388714	3	4566120917
5	104	966352962	91	2712096960
7	264	1230936742	187	2610072986
11	472	1232751553	403	2323241429
13	872	1198845428	763	1938143851
17	1464	1025781633	1243	1562298824
19	1672	887051278	3243	1187742265
23	1992	743252807	3235	907034651
29	5272	570122741	7483	672787920
31	4888	440661611	5707	493897409

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Table 14 – continued from previous page

p	first even Δ	# even Δ	first odd Δ	# odd Δ
37	6232	325713241	14155	359012330
41	21172	241825471	16867	266476682
43	13960	187418933	25267	189448477
47	21912	136442158	16003	138707810
53	18232	98485029	41827	98465622
59	25048	71667859	52627	69185532
61	26440	52768397	85507	49599909
67	121972	37698631	30067	35147411
71	50152	27170708	133827	25110146
73	77928	19454376	130123	17845045
79	180552	14075585	111763	12257594
83	249208	9839005	282027	8977750
89	371508	6990894	232243	6141293
97	340312	5027158	383667	4301305
101	513832	3694362	1133587	3054505
103	652488	2578330	347755	2108368
107	300568	1813002	1477387	1540587
109	815028	1273170	5103267	1032731
113	2402328	891724	2462835	735704
127	3699172	641658	1852547	498538
131	424708	449660	3466803	356461
137	6649080	312663	13574595	250697
139	2764248	218769	17601987	168520
149	1826248	157522	32872107	119230
151	5580568	112181	9486555	78834
157	14411832	78374	34378323	55071
163	11268632	52425	42132435	38170
167	32708760	37839	47696907	27146
173	22824120	25231	93649147	18227
179	131619412	17709	73224715	12596
181	144539192	12587	314004243	8575
191	102897960	8467	320578003	6074
193	222843252	6136	176606203	3817
197	38772728	4167	745996867	2877
199	126216456	2917	723521635	1828
211	113842920	2075	860305747	1222
223	1097298568	1306	385977883	856
227	1090031860	938	393858987	661
229	1374468472	640	965986483	387
233	770781768	491	3773818003	301
239	1123437128	308	2077328467	216
241	1536253048	226	428548387	134
251	2289587848	164	1734872043	104
257	4866362920	103	11444879803	77
263	8452497652	85	2812125387	34

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Table 14 – continued from previous page

p	first even Δ	# even Δ	first odd Δ	# odd Δ
269	6746496232	57	20850840627	24
271	5091936072	30	7677719107	22
277	4613771208	19	8019330403	23
281	28218429460	10	59103258547	3
283	28817461272	11	22797263523	7
293	15639715092	6	28343168643	6
307	51185074312	3	59197844315	3
311	39081740072	2	39175329139	6
313	29696961688	3	*	*
317	*	*	*	*
331	29307259048	1	37228840027	1
337	42964341688	1	*	*
347	*	*	66324417027	1
349	91500037960	1	32920214803	1
353	88460711448	1	42930759883	1

Table 15. First Δ needing k prime ideals

k	first even Δ	# even Δ	first odd Δ	# odd Δ
0	4	2	3	7
1	20	270754320	15	3946412653
2	68	1686719293	119	4934710984
3	264	2574927846	759	4746914671
4	888	2339302287	3615	3282922437
5	4980	1620705305	9867	1775780537
6	13960	868450546	19635	887769165
7	26440	428785203	107835	401785772
8	124440	194939831	498355	171948318
9	320712	86444646	1001715	71671258
10	563640	37099441	3674715	28482310
11	2673528	15156252	12633027	10576275
12	5053620	5760153	13735995	3626603
13	18038020	2059679	42082755	1160816
14	37612840	697042	156515755	347151
15	139350660	223400	384745155	95707
16	222843252	67511	582274555	24644
17	569046520	18913	1658763555	5877
18	608451288	4940	2060262283	1269
19	1374468472	1279	4003220067	267
20	6746496232	317	14002238827	56
21	2886861432	72	30892011195	9
22	22870805160	12	37228840027	1
23	71409801540	1	*	*

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Table 15 – continued from previous page

k	first even Δ	# even Δ	first odd Δ	# odd Δ
24	*	*	*	*
25	75948116920	1	*	*

Table 16. Values of $\max p/\log \Delta$ and $\max p/\log^2 \Delta$

x	$\max p/\log \Delta$			$\max p/\log^2 \Delta$		
	ave	max	Δ	*ave*	max	Δ
1000000000	0.66488	12.12523	428548387	0.03385	0.78004	424708
2000000000	0.65472	12.12523	428548387	0.03220	0.78004	424708
3000000000	0.64888	12.12523	428548387	0.03128	0.78004	424708
4000000000	0.64479	12.12523	428548387	0.03066	0.78004	424708
5000000000	0.64164	12.44815	4613771208	0.03019	0.78004	424708
6000000000	0.63909	12.44815	4613771208	0.02981	0.78004	424708
7000000000	0.63694	12.44815	4613771208	0.02950	0.78004	424708
8000000000	0.63510	12.44815	4613771208	0.02923	0.78004	424708
9000000000	0.63347	12.44815	4613771208	0.02900	0.78004	424708
10000000000	0.63203	12.44815	4613771208	0.02879	0.78004	424708
20000000000	0.62267	12.48238	15639715092	0.02750	0.78004	424708
30000000000	0.61731	13.73381	29307259048	0.02678	0.78004	424708
40000000000	0.61356	14.41115	32920214803	0.02629	0.78004	424708
50000000000	0.61068	14.41825	42930759883	0.02592	0.78004	424708
60000000000	0.60835	14.41825	42930759883	0.02562	0.78004	424708
70000000000	0.60639	14.41825	42930759883	0.02537	0.78004	424708
80000000000	0.60471	14.41825	42930759883	0.02516	0.78004	424708
90000000000	0.60323	14.41825	42930759883	0.02497	0.78004	424708
100000000000	0.60191	14.41825	42930759883	0.02481	0.78004	424708