Computing Prime Harmonic Sums

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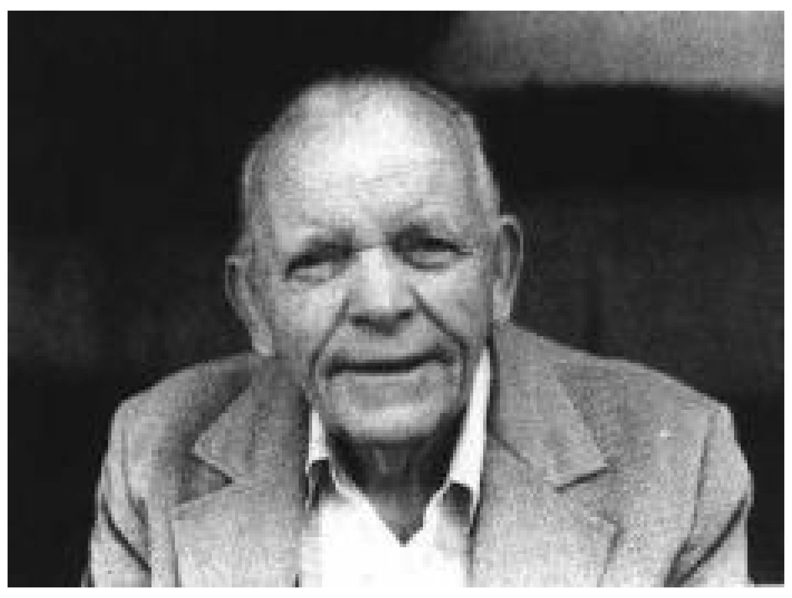
Overview

We present an algorithm for computing

$$S(x) := \sum_{p \le x} \frac{1}{p}$$

using $x^{2/3+o(1)}$ time and $x^{1/3+o(1)}$ space. Our algorithm is based on the Meissel-Lehmer algorithm for computing the prime-counting function $\pi(x)$, which was adapted and improved by Lagarias, Miller, and Odlyzko [2, 6, 7].

E. D. F. Meissel



D. H. Lehmer

Basic Formulas

Let p_i denote the ith prime, and let $\ell(n)$ denote the least prime factor of n. We define

$$\phi(x,a) := \sum_{\substack{n \le x \\ \ell(n) > p_a}} \frac{1}{n}.$$

Let S_k be this same sum, with n restricted to precisely k prime factors, with $S_0 = 1$, so that

$$\phi(x,a) = S_0 + S_1 + S_2 + S_3 + \cdots .$$

Now we set $a := \pi(x^{1/3})$, so that p_a is the largest prime $\le x^{1/3}$, making $S_k = 0$ for $k > 2$. Observe that

$$S_1 = \sum_{p \le x} \frac{1}{p} - \sum_{p \le p_a} \frac{1}{p} \quad \text{and} \quad S_2 = \sum_{\substack{pq \le x \\ p_a < p \le q}} \frac{1}{pq}.$$

This leads us to the following formula for $S(x)$:

$$\sum_{p \le x} \frac{1}{p} = \sum_{p \le p_a} \frac{1}{p} + \phi(x,a) - 1 - \sum_{\substack{pq \le x \\ p_a < p \le q}} \frac{1}{pq}. \qquad (1)$$

Algorithm Outline

1. The first term in (1) can be computed in $x^{1/3}$ time using a prime sieve. This is the dominant term, asymptotic to $\log \log x$ by (2).
2. For $x, a \ge 1$ we have

   $$\phi(x,-1) = \phi(x,a) + \frac{1}{p_a} \phi(x/p_a, a-1)$$

   which implies that the second term in (1), $\phi(x,a)$, satisfies the recurrence relation

   $$\phi(x,a) = \phi(x,a-1) - \phi(x/p_a, a-1)/p_a.$$

   From this, we are able to show

   $$\phi(x,a) = \sum_{(m,b) \text{ ordinary}} \frac{\mu(m)}{m} \phi(x/m,b) + \sum_{(m,b) \text{ special}} \frac{\mu(m)}{m} \phi(x/m,b),$$

   where $(x/m, b)$ is either *ordinary* if $b = 1$ and $m \le x^{1/3}$, or *special* if $m > x^{1/3}$ (and never both). Using segmented sieving together with a special tree data structure for computing range sums [3], $\phi(x,a)$ can be computed in $x^{2/3+o(1)}$ time and $x^{1/3+o(1)}$ space.
3. The last term in (1) can be rewritten as

   $$\sum_{\substack{pq \le x \\ p_a < p \le q}} \frac{1}{pq} = \sum_{p_a < p \le \sqrt{x}} \frac{1}{p} \left[\sum_{q \le x/p} \frac{1}{q} - \sum_{q \le p} \frac{1}{q} + \frac{1}{p} \right]$$

   which permits us to evaluate it in $x^{2/3+o(1)}$ time.

A Computation Inspired by Neal Sloane

Neal Sloane asked for the smallest value of x such that $S(x)$ exceeds 4. Mertens's Second Theorem [5, §22.7] states that

$$\sum_{p \le x} \frac{1}{p} = \log \log x + B + O\left(\frac{1}{\log x} \right), \qquad (2)$$

where

$$B = \gamma + \sum_p \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) = 0.26149721 \ldots .$$

By assuming the Riemann Hypothesis, Schoenfeld [8] proved the following explicit version of (2):

$$\left| \sum_{p \le x} \frac{1}{p} - \log \log x - B \right| < \frac{3 \log x + 4}{8 \pi \sqrt{x}}$$

when $x \ge 13.5$. From this we obtain the estimate that when $S(x)$ first exceeds 4 we have

$$1.80124093... \times 10^{18} < x < 1.80124152... \times 10^{18},$$

the *Schoenfeld interval*. We computed $S(x)$ for

$$x = 1216720^3 = 1801241484456448000,$$

and then used a precomputed table of sums in the Schoenfeld interval (thereby avoiding interpolation or binary search and multiple evaluations of $S(x)$) to discover

$$S(1801241230056600467) \le 3.99999\ 99999\ 99999\ 99966 \quad \text{and}$$
$$S(1801241230056600523) \ge 4.00000\ 00000\ 00000\ 00021.$$

There are no primes between these two x values.
This computation took roughly one week on two workstations.

Franz Mertens

Josephine Mitchell
& Lowell Schoenfeld

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