Computing Prime Harmonic Sums
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Overview

In this poster we will present an algorithm for computing $S(x) := \sum_{p \le x} 1/p$ using $x^{2/3+o(1)}$ time and $x^{1/3+o(1)}$ space. Our algorithm is based on the Meissel-Lehmer algorithm for computing the prime-counting function $\pi(x)$, which was adapted and improved by Lagarias, Miller, and Odlyzko [2, 6, 7].

Algorithm Outline

Let p_i denote the ith prime, and let p_a be the largest prime $\le x^{1/3}$ (that is, $a = \pi(x^{1/3})$). Then we can show

$$S(x) := \sum_{p \le x} \frac{1}{p} \quad = \quad \sum_{p \le p_a} \frac{1}{p} + \phi(x, a) - 1 - \sum_{\substack{pq \le x \\ p_a < p \le q}} \frac{1}{pq}, \tag{1}$$

where $\phi(x, a) := \sum_{n \le x, \ell(n) > p_a} (1/n)$, and $\ell(n)$ is the least prime factor of n.

1. The first term in (1) can be computed in $x^{1/3}$ time using a prime sieve. This is the dominant term, asymptotic to $\log \log x$.
2. The last term can be rewritten as
$$\sum_{p_a < p \le \sqrt{x}} \frac{1}{p} \left[\sum_{q \le x/p} \frac{1}{q} - \sum_{q \le p} \frac{1}{q} + \frac{1}{p} \right]$$
which permits us to evaluate it in $x^{2/3+o(1)}$ time.
3. The second term, $\phi(x, a)$, satisfies the recurrence relation $\phi(x, a) = \phi(x, a-1) - \phi(x/p_a, a-1)/p_a$. From this, we show
$$\phi(x, a) = \sum_{(m,b) \text{ ordinary}} \frac{\mu(m)}{m} \phi(x/m, b) + \sum_{(m,b) \text{ special}} \frac{\mu(m)}{m} \phi(x/m, b),$$
where $(x/m, b)$ is either *ordinary* if $b = 1$ and $m \le x^{1/3}$, or *special* if $m > x^{1/3}$ (and never both). Using segmented sieving together with a special tree datastructure for computing range sums [3], $\phi(x, a)$ can be computed in $x^{2/3+o(1)}$ time and $x^{1/3+o(1)}$ space.

A Computation

We used our algorithm to find the smallest value x for which $S(x)$ exceeds 4. Using an explicit error estimate due to Schoenfeld [11] based on the Riemann Hypothesis, we have the estimate

$$1.80124093... \times 10^{18} < x < 1.80124152... \times 10^{18}.$$

We computed $S(x)$ for $x = 1216720^3 = 1801241484456448000$, and then used a precomputed table of sums in the Schoenfeld interval (thereby avoiding interpolation or binary search and multiple evaluations of $S(x)$) to discover

$$S(1801241230056600467) \le 3.99999\ 99999\ 99999\ 99966 \qquad \text{and}$$
$$S(1801241230056600523) \ge 4.00000\ 00000\ 00000\ 00021.$$

There are no primes between these two x values. This computation took roughly 1 week on two workstations.

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