# The Carmichael numbers up to $10^{20}$

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## Introduction

We extend our previous computations to show that there are 8220777 Carmichael numbers up to $10^{20}$. As before, the numbers were generated by a back-tracking search for possible prime factorisations together with a "large prime variation". We present further statistics on the distribution of Carmichael numbers.

## Organisation of the search

We used improved versions of strategies first described in [2].

The principal search was a depth-first back-tracking search over possible sequences of primes factors $p_1,\dots,p_d$. Put $P_r = \prod_{i=1}^r p_i$, $Q_r = \prod_{i=r+1}^d p_i$ and $L_r = \mathrm{lcm}\{p_i - 1 : i = 1,\dots,r\}$. We find that $Q_r$ must satisfy the congruence $N = P_r Q_r \equiv 1 \bmod L_r$ and so in particular $Q_d = p_d$ must satisfy a congruence modulo $L_{d-1}$: further $p_d - 1$ must be a factor of $P_{d-1} - 1$. We modified this to terminate the search early at some level $r$ if the modulus $L_r$ is large enough to limit the possible values of $Q_r$, which may then be factorised directly.

We also employed the variant based on proposition 2 of [2] which determines the finitely many possible pairs $(p_{d-1}, p_d)$ from $P_{d-2}$. In practice this was useful only when $d = 3$ allowing us to determine the complete list of Carmichael numbers with three prime factors up to $10^{20}$.

### A large prime variation

Finally we employed a different search over large values of $p_d$, in the range $2.10^6 < p_d < 10^{9.5}$, using the property that $P_{d-1} \equiv 1 \bmod (p_d - 1)$.

If $q$ is a prime in this range, we let $P$ run through the arithmetic progression $P \equiv 1 \bmod q-1$ in the range $q < P < X/q$ where $X = 10^{20}$. We first check whether $N = Pq$ satisfies $2^N \equiv 2 \bmod N$: it is sufficient to test whether $2^N \equiv 2 \bmod P$ since the congruence modulo $q$ is necessarily satisfied. If this condition is satisfied we factorise $P$ and test whether $N \equiv 1 \bmod \lambda(N)$.

The approximate time taken for $X^t \le q < X^{1/2}$ is

$$\sum_{X^t < q < X^{1/2}} \frac{X}{q^2} \approx X^{1-t}.$$

## Statistics

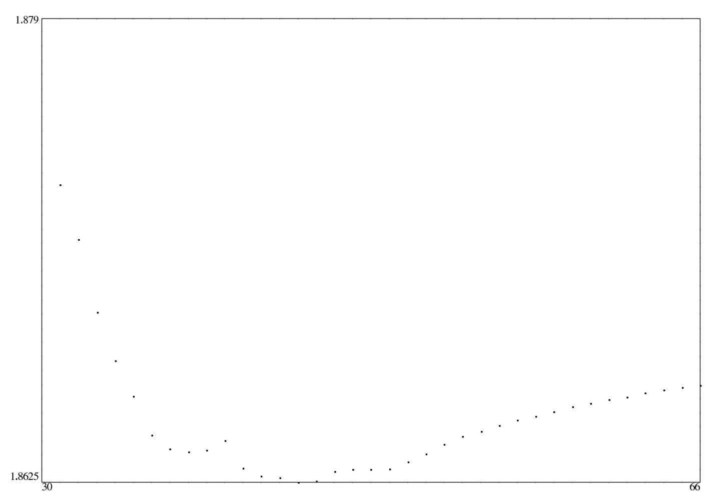
| $n$ | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| $C(n)$ | 8241 | 19279 | 44706 | 105212 | 246683 | 585355 | 1401644 | 3381806 | 8220777 |

*Distribution of Carmichael numbers up to $10^{19}$.*

| $X$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 4 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 |
| 5 | 12 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 |
| 6 | 23 | 19 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 43 |
| 7 | 47 | 55 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 105 |
| 8 | 84 | 144 | 27 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 |
| 9 | 172 | 314 | 146 | 14 | 0 | 0 | 0 | 0 | 0 | 0 | 646 |
| 10 | 335 | 619 | 492 | 99 | 2 | 0 | 0 | 0 | 0 | 0 | 1547 |
| 11 | 590 | 1179 | 1336 | 459 | 41 | 0 | 0 | 0 | 0 | 0 | 3605 |
| 12 | 1000 | 2102 | 3156 | 1714 | 262 | 7 | 0 | 0 | 0 | 0 | 8241 |
| 13 | 1858 | 3639 | 7082 | 5270 | 1340 | 89 | 1 | 0 | 0 | 0 | 19279 |
| 14 | 3284 | 6042 | 14938 | 14401 | 5359 | 655 | 27 | 0 | 0 | 0 | 44706 |
| 15 | 6083 | 9938 | 29282 | 36907 | 19210 | 3622 | 170 | 0 | 0 | 0 | 105212 |
| 16 | 10816 | 16202 | 55012 | 86696 | 60150 | 16348 | 1436 | 23 | 0 | 0 | 246683 |
| 17 | 19539 | 25758 | 100707 | 194306 | 172234 | 63635 | 8835 | 340 | 1 | 0 | 585355 |
| 18 | 35586 | 40685 | 178063 | 414660 | 460553 | 223997 | 44993 | 3058 | 49 | 0 | 1401644 |
| 19 | 65309 | 63343 | 306310 | 849564 | 1159167 | 720406 | 196391 | 20738 | 576 | 2 | 3381806 |
| 20 | 1250625 | 98253 | 514381 | 1681744 | 2774702 | 2148017 | 762963 | 114232 | 5804 | 56 | 8220777 |

*Values of $C(X)$ and $C(d,X)$ for $d \le 10$ and $X$ in powers of 10 up to $10^{20}$.*

We have shown that there are 8220777 Carmichael numbers up to $10^{20}$, all with at most 12 prime factors. We let $C(X)$ denote the number of Carmichael numbers less than $X$ and $C(d,X)$ denote the number with exactly $d$ prime factors. Table 1 gives the values of $C(X)$ and Table 2 the values of $C(d,X)$ for $X$ in powers of 10 up to $10^{20}$.



*$k(X)$ versus $\log_2 X$.*

In Table 3 and Figure 1 we tabulate the function $k(X)$, defined by Pomerance, Selfridge and Wagstaff [3] by

$$C(X) = X \exp\left(-k(X)\frac{\log X \log\log\log X}{\log\log X}\right).$$

They proved that $\liminf k \ge 1$ and suggested that $\limsup k$ might be 2, although they also observed that within the range of their tables $k(X)$ is decreasing: Pomerance [4],[5] gave a heuristic argument suggesting that $\lim k = 1$. The decrease in $k$ is reversed between $10^{13}$ and $10^{14}$: see Figure 1. We find no clear support from our computations for any conjecture on a limiting value of $k$.

| $n$ | $\log C(10^n)/(n \log 10)$ | $C(10^n)/C(10^{n-1})$ | $k(10^n)$ |
|---|---|---|---|
| 4 | 0.21127 | 7.000 | 2.19547 |
| 5 | 0.24082 | 2.286 | 2.07632 |
| 6 | 0.27224 | 2.688 | 1.97946 |
| 7 | 0.28874 | 2.441 | 1.93388 |
| 8 | 0.30082 | 2.429 | 1.90495 |
| 9 | 0.31225 | 2.533 | 1.87989 |
| 10 | 0.31895 | 2.396 | 1.86870 |
| 11 | 0.32336 | 2.330 | 1.86421 |
| 12 | 0.32633 | 2.286 | 1.86377 |
| 13 | 0.32962 | 2.339 | 1.86240 |
| 14 | 0.33217 | 2.319 | 1.86293 |
| 15 | 0.33480 | 2.353 | 1.86301 |
| 16 | 0.33700 | 2.335 | 1.86406 |
| 17 | 0.33926 | 2.373 | 1.86472 |
| 18 | 0.34148 | 2.394 | 1.86522 |
| 19 | 0.34363 | 2.413 | 1.86565 |
| 20 | 0.34574 | 2.431 | 1.86598 |

*$C(X)$ as a power of $X$, the growth of $C(X)$ and the function $k(X)$ for $X = 10^n$ up to $10^{20}$.*

## Conclusion

We consider that there is no clear evidence that $k(x)$ approaches any limit.

## References

[1] R.A. Mollin (ed.), *Number theory and its applications*, Dordrecht, Kluwer Academic, 1989, Proceedings of the NATO Advanced Study Institute on Number Theory and Applications.

[2] Richard G.E. Pinch, *The Carmichael numbers up to* $10^{15}$, Math. Comp. **61** (1993), 381–391, Lehmer memorial issue.

[3] C. Pomerance, J.L. Selfridge, and S.S. Wagstaff jr, *The pseudoprimes up to* $25.10^9$, Math. Comp. **35** (1980), no. 151, 1003–1026.

[4] Carl Pomerance, *On the distribution of pseudoprimes*, Math. Comp. **37** (1981), 587–593.

[5] ______, *Two methods in elementary analytic number theory*, in Mollin [1], Proceedings of the NATO Advanced Study Institute on Number Theory and Applications, pp. 136–161.