Elliptic curves with good reduction away from 2

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Introduction

We list the elliptic curves defined over $\mathbb{Q}(\sqrt{5})$ with good reduction away from 2. There are 368 isomorphism classes.

Equations with good reduction away from 2

Table 1 lists values of t corresponding to a curve with good reduction away from 2, together with the corresponding values of the modular invariant j and their factorisations.

t		*code*	*j*	
$-220+136\varepsilon$	$+\varepsilon^{-9}2^2$	34	78608	$+2^4(17)^3$
$-84-136\varepsilon$	$-\varepsilon^{9}2^2$	34	78608	$+2^4(17)^3$
$-64+40\varepsilon$	$+\varepsilon^{-5}2^3$	28	$4928+960\varepsilon$	$+\varepsilon^{2}2^6(5-\varepsilon)^3$
$40-24\varepsilon$	$+\varepsilon^{-4}2^3$	31	$5888-960\varepsilon$	$+\varepsilon^{-2}2^6(4+\varepsilon)^3$
8	$+2^3$	11	1728	$+2^6(3)^3$
$16+24\varepsilon$	$+\varepsilon^{4}2^3$	28	$4928+960\varepsilon$	$+\varepsilon^{2}2^6(5-\varepsilon)^3$
$-24-40\varepsilon$	$-\varepsilon^{5}2^3$	31	$5888-960\varepsilon$	$+\varepsilon^{-2}2^6(4+\varepsilon)^3$
$-48+32\varepsilon$	$+\varepsilon^{-3}2^4$	14	2048	$+2^{11}$
16	$+2^4$	14	2048	$+2^{11}$
$-16-32\varepsilon$	$-\varepsilon^{3}2^4$	14	2048	$+2^{11}$
$-19520+12064\varepsilon$	$-\varepsilon^{-14}2^5$	40	$525604480-324907648\varepsilon$	$-\varepsilon^{-7}2^7(50-13\varepsilon)^3$
$256-160\varepsilon$	$-\varepsilon^{-5}2^5$	2	$103808-64640\varepsilon$	$-\varepsilon^{-7}2^7(4+\varepsilon)^3$
$-64+32\varepsilon$	$-\varepsilon^{-2}2^5$	17	$2688-1664\varepsilon$	$-\varepsilon^{-7}2^7$
$-32+32\varepsilon$	$+\varepsilon^{-1}2^5$	6	$1280+640\varepsilon$	$-\varepsilon 2^7(1-2\varepsilon)^3$
-32	-2^5	5	128	$+2^7$
-32ε	$-\varepsilon 2^5$	13	$1920-640\varepsilon$	$-\varepsilon^{-1}2^7(1-2\varepsilon)^3$
$-32-32\varepsilon$	$-\varepsilon^{2}2^5$	1	$1024+1664\varepsilon$	$+\varepsilon^{7}2^7$
$96+160\varepsilon$	$+\varepsilon^{5}2^5$	23	$39168+64640\varepsilon$	$+\varepsilon^{7}2^7(5-\varepsilon)^3$
$-7456-12064\varepsilon$	$-\varepsilon^{14}2^5$	41	$200696832+324907648\varepsilon$	$+\varepsilon^{7}2^7(37+13\varepsilon)^3$
$-832+512\varepsilon$	$-\varepsilon^{-6}2^6$	39	$914880-565760\varepsilon$	$+\varepsilon^{-12}2^6(-1-8\varepsilon)^3$
$192-128\varepsilon$	$-\varepsilon^{-3}2^6$	32	$63168-39040\varepsilon$	$-\varepsilon^{-15}2^6$
$-192+128\varepsilon$	$+\varepsilon^{-3}2^6$	26	$44864-26496\varepsilon$	$+\varepsilon^{-6}2^6(5+2\varepsilon)^3$
$-128+64\varepsilon$	$-\varepsilon^{-2}2^6$	8	$15040-9280\varepsilon$	$-\varepsilon^{-7}2^6(1-2\varepsilon)^3$
$64-64\varepsilon$	$-\varepsilon^{-1}2^6$	4	$12032-7232\varepsilon$	$+\varepsilon^{-8}2^6(3+2\varepsilon)^3$
$-64+64\varepsilon$	$+\varepsilon^{-1}2^6$	31	$5888-960\varepsilon$	$+\varepsilon^{-2}2^6(4+\varepsilon)^3$
-64	-2^6	11	1728	$+2^6(3)^3$
64	$+2^6$	35	8000	$+2^6(5)^3$
-64ε	$-\varepsilon 2^6$	28	$4928+960\varepsilon$	$+\varepsilon^{2}2^6(5-\varepsilon)^3$
64ε	$+\varepsilon 2^6$	27	$4800+7232\varepsilon$	$+\varepsilon^{8}2^6(5-2\varepsilon)^3$
$-64-64\varepsilon$	$-\varepsilon^{2}2^6$	30	$5760+9280\varepsilon$	$-\varepsilon^{7}2^6(1-2\varepsilon)^3$
$-64-128\varepsilon$	$-\varepsilon^{3}2^6$	12	$18368+26496\varepsilon$	$+\varepsilon^{6}2^6(7-2\varepsilon)^3$
$64+128\varepsilon$	$+\varepsilon^{3}2^6$	16	$24128+39040\varepsilon$	$+\varepsilon^{15}2^6$
$-320-512\varepsilon$	$-\varepsilon^{6}2^6$	22	$349120+565760\varepsilon$	$-\varepsilon^{12}2^6(9-8\varepsilon)^3$
$-78080+48256\varepsilon$	$-\varepsilon^{-14}2^7$	42	$8421373408-5204711200\varepsilon$	$+\varepsilon^{-13}2^5(-85+6\varepsilon)^3$
$-1024+640\varepsilon$	$+\varepsilon^{-5}2^7$	7	$1409888-870240\varepsilon$	$+\varepsilon^{-4}2^5(17-6\varepsilon)^3$
$-256+128\varepsilon$	$-\varepsilon^{-2}2^7$	33	$70368-43040\varepsilon$	$+\varepsilon^{-4}2^5(7-\varepsilon)^3$
$128-128\varepsilon$	$-\varepsilon^{-1}2^7$	24	$39680-22560\varepsilon$	$-\varepsilon 2^5(9-8\varepsilon)^3$
-128	-2^7	3	10976	$+2^5(7)^3$
128ε	$+\varepsilon 2^7$	10	$17120+22560\varepsilon$	$-\varepsilon^{-1}2^5(-1-8\varepsilon)^3$
$-128-128\varepsilon$	$-\varepsilon^{2}2^7$	18	$27328+43040\varepsilon$	$+\varepsilon^{4}2^5(6+\varepsilon)^3$
$-384-640\varepsilon$	$-\varepsilon^{5}2^7$	29	$539648+870240\varepsilon$	$+\varepsilon^{4}2^5(11+6\varepsilon)^3$
$-29824-48256\varepsilon$	$-\varepsilon^{14}2^7$	43	$3216662208+5204711200\varepsilon$	$-\varepsilon^{13}2^5(-79-6\varepsilon)^3$
$768-512\varepsilon$	$-\varepsilon^{-3}2^8$	36	$889584-548896\varepsilon$	$+\varepsilon^{-6}2^4(15-2\varepsilon)^3$
256	$+2^8$	34	78608	$+2^4(17)^3$
$256+512\varepsilon$	$+\varepsilon^{3}2^8$	20	$340688+548896\varepsilon$	$+\varepsilon^{6}2^4(13+2\varepsilon)^3$
$4096-2560\varepsilon$	$-\varepsilon^{-5}2^9$	15	$23528168-14540840\varepsilon$	$-\varepsilon^{-10}2^3(-29+5\varepsilon)^3$
$2560-1536\varepsilon$	$+\varepsilon^{-4}2^9$	38	$9036560-5578728\varepsilon$	$-\varepsilon^{-8}2^3(-34-3\varepsilon)^3$
512	$+2^9$	19	287496	$+2^3(33)^3$
$1024+1536\varepsilon$	$+\varepsilon^{4}2^9$	21	$3457832+5578728\varepsilon$	$-\varepsilon^{8}2^3(-37+3\varepsilon)^3$
$1536+2560\varepsilon$	$+\varepsilon^{5}2^9$	37	$8987328+14540840\varepsilon$	$-\varepsilon^{10}2^3(-24-5\varepsilon)^3$
$56320-34816\varepsilon$	$-\varepsilon^{-9}2^{10}$	25	$4386800300-2711191688\varepsilon$	$-\varepsilon^{-21}2^2(34-9\varepsilon)^3$
$21504+34816\varepsilon$	$+\varepsilon^{9}2^{10}$	9	$1675608612+2711191688\varepsilon$	$+\varepsilon^{21}2^2(25+9\varepsilon)^3$

Invariants of elliptic curves defined over $\mathbb{Q}\left(\sqrt{5}\right)$ with good reduction away from 2

Theorem 1. *There are 56 values of t and 43 values of j corresponding to elliptic curves over $\mathbb{Q}(\sqrt{5})$ with good reduction away from 2. There are 368 isomorphism classes of such curves.*

It is interesting to observe that every curve in has additive bad reduction. In partcular, we note that there is no curve with good reduction everywhere over $\mathbb{Q}(\sqrt{5})$: see Cremona [2], Ishii [3],[4] and Pinch [6].

Diophantine equations over the quadratic field

We use the results of the previous papers [7] and [8], referred to as I and II respectively. By Theorem 1.14 of I, such a curve must have a point of order 2 defined over $\mathbb{Q}(\sqrt{5})$ and by Theorem 2.3 of II, if $t \in \mathbb{Q}(\sqrt{5})$ is the corresponding value of the Hauptmodul on $X_0(2)$ then either t or $t' = 4096/t$ satisfies one of the equations

$$t = 64u/v, \quad u+v = x^2 \qquad (1)$$

$$t = 64v/2^a u, \quad 2^a u + v = x^2 \qquad (2)$$

where u, v are units, $x \in \mathbb{Q}(\sqrt{5})$ and $a \geq 0$.

We solve the equations (1) and (2) over $\mathbb{Q}(\sqrt{5})$. The fundamental unit is $\varepsilon = \frac{1+\sqrt{5}}{2}$ and the ring of integers is $\mathbb{Z}[\varepsilon]$. Let $'$ denote the non-trivial automorphism of $\mathbb{Q}(\sqrt{5})$.

The proofs are mainly elementary: we cite Cohn [1] for some cases. The final case requires a practical application of transcendence methods, described in [9] and further further developed in [10].

Theorem 2. *The equation*

$$2x^2 = u+v, \qquad u, v \text{ units of } \mathbb{Q}(\sqrt{5}),$$

has solutions

	x	u	v
a)	0	1	-1
b)	1	1	1
c)	1	ε^2	ε'
d)	3	ε^6	ε'^6

and all other solutions are obtained by conjugation and scaling by units.

The equations

$$x^2 = 2^a u + v, \qquad u, v \text{ units}, \quad a \geq 0,$$

have solutions

	a	x	u	v
e)	0	0	1	-1
f)	0	1	ε	ε'
g)	0	1	ε^2	$-\varepsilon$
h)	0	2	ε^3	ε'^3
i)	1	1	1	-1
j)	1	1	$-\varepsilon$	ε^3
k)	1	ε	1	$-\varepsilon'$
l)	1	$\sqrt{5}$	ε^2	ε'^3
m)	1	$8+15\varepsilon$	ε^{13}	ε'
n)	2	$\sqrt{5}$	1	1
o)	2	ε^3	ε^3	1
p)	3	3	1	1
q)	3	$5+2\sqrt{5}$	ε^5	1
r)	3	$3+2\sqrt{5}$	ε^4	1
s)	4	$17+8\sqrt{5}$	ε^9	1

and all other solutions are obtained by conjugation or scaling by units.

The corresponding values of t are

a,e) -64 *b)* 64 *c)* $-64\varepsilon^3$ *d,h)* $-64\varepsilon^6$ *f)* $-64\varepsilon^2$ *g)* 64ε
i) -32 *j)* -32ε *k)* $-32\varepsilon^2$ *l)* $32\varepsilon^5$ *m)* $-32\varepsilon^{14}$ *n)* 16
o) $-16\varepsilon^3$ *p)* 8 *q)* $-8\varepsilon^5$ *r)* $8\varepsilon^4$ *s)* $-4\varepsilon^9$

and their conjugates.

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