University of Waterloo

Linear Forms in Logarithms and Thue Equations

Robert Juricevic, Pure Mathematics Department, University of Waterloo
Waterloo, Ontario, Canada

Objects of Study

Linear Forms in Logs

$$\sum_{i=1}^{n} b_i \log \alpha_i$$

Thue Equations

$$\sum_{i=0}^{n} a_i x^i y^{n-i} = m$$

The Mordell Equation

$$y^2 - x^3 = k$$

Lehmer Sequences

$$\left(\frac{\alpha^i - \beta^i}{\alpha^\delta - \beta^\delta} \right)_{i=0}^{\infty} \quad \delta = \begin{cases} 1, i \equiv 1 \bmod 2 \\ 2, i \equiv 0 \bmod 2 \end{cases}$$

Problems of Study

1. With respect to the Mordell equation, establish an explicit upper bound on the maximum of the absolute value of x and of the absolute value of y as a function of the absolute value of k.

2. With respect to Lehmer sequences, classify all Lehmer triples (n,α,β) such that (α,β) is a Lehmer pair, and the n^{th} term of the Lehmer sequence has r primitive divisor(s), where r is a non-negative integer.

Theoretical Results

Extending his ground breaking work on Hilbert's 10^{th} problem, and in particular on Thue equations, to the Mordell equation, Baker, with respect to problem 1, was the first to establish that for any non zero integer k, all solutions of the Mordell equation in integers x and y satisfy

$$\log \max \{|x|, |y|\} < c_1 |k|^{10,000} .$$

We show that the tools now available in the literature imply

$$\log \max \{|x|, |y|\} < |k| (\log |k|)^4 \cdot \min \{c_2, c_3 \log |k|, c_4 (\log |k|)^2\}$$

Note that the implied constants $c_1>c_2>c_3>c_4$ are precise positive integers not specified here.

Computational Results

Extending the explicit work of Stewart on primitive divisors of terms of Lehmer sequences, Bilu, Hanrot, Voutier and Mignotte, astonishingly solved problem 2 in case r=0.

Among other results, in case r=1 in problem 2 we extend Schinzel's work and determine explicitly the finite set of all Lehmer triples (n,α,β), ignoring here conditions on n, such that (α,β) is a real Lehmer pair, and the n^{th} term of the Lehmer sequence has 1 primitive divisor.

We list a sample of this set below. We omit the 0^{th}, 1^{st}, and 2^{nd} term of the Lehmer sequence in our listing below. We benefited from the calculators WATERLOO MAPLE and KASH/KANT in our work.

(n, α,β) , [Lehmer Sequence], $(\alpha^2-\beta^2)^2$

$$\left(5, \frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}-1}{2}\right), [4, 3, 11, \ldots], 5$$

$$\left(5, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right), [8, 7, 5 \cdot 11, \ldots], 3^2 \cdot 5$$

$$\left(7, \frac{\sqrt{3}+\sqrt{7}}{2}, \frac{\sqrt{3}-\sqrt{7}}{2}\right), [4, 5, 19, 24, 7 \cdot 13, \ldots], 3 \cdot 7$$

$$\left(10, \frac{\sqrt{5}+1}{2}, \frac{1-\sqrt{5}}{2}\right), [2, 3, 5, 8, 13, 21, 34, 5 \cdot 11, \ldots], 5$$

$$\left(10, \frac{3+\sqrt{5}}{2}, \frac{\sqrt{5}-3}{2}\right), [6, 7, 41, 48, 281, 329, 1926, 5 \cdot 11 \cdot 41, \ldots], 3^2 \cdot 5$$

$$(12, 3, -2), [7, 13, 55, 7 \cdot 19, 463, 13 \cdot 97, 4039, 11605, 35839, 7 \cdot 13 \cdot 19 \cdot 61, \ldots], 25$$

$$(12, 2, -1), [3, 5, 11, 3 \cdot 7, 43, 85, 171, 341, 683, 3 \cdot 5 \cdot 7 \cdot 13, \ldots], 9$$

References

1. **Baker, A.,** *Contributions to the Theory of Diophantine Equations II*, Phil. Trans. Lond. Math. Soc. Ser. A, 1967.
2. **Bilu, Hanrot, Voutier and Mignotte,** *Existence of primitive divisors of Lucas and Lehmer numbers,* J.Reine Angew, 2001.
3. **Pohst, M.** et al, *KANT V4,* J. Symbolic Computation, 1997.
4. **Schinzel, A.,** *On primitive prime factors of Lehmer Numbers I*, Acta. Arith. VIII, 1963.
5. **Stewart, C.L.,** *Primitive divisors of Lucas and Lehmer numbers*, Trans. Num. Theory, Academic Press, 1977.

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