Computing automorphic forms on Shimura curves over fields with arbitrary class number

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$N\mathfrak{p}$	π	$a(\mathfrak{p})$
2	$w - 2$	0
3	$w - 3$	-3
3	$w - 1$	-1
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There exists a (modular!) elliptic curve J over F such that $\#J(\mathbb{F}_\mathfrak{p}) = N\mathfrak{p} + 1 - a(\mathfrak{p})$...

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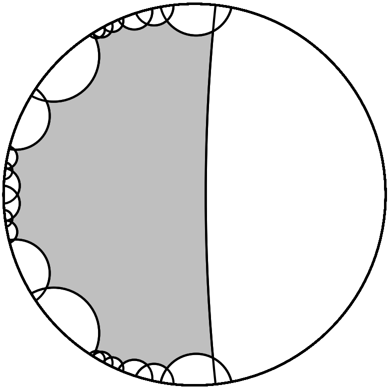

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$$A : y^2 + w(w+1)xy + (w+1)y = x^3 + w^2x^2 + a_4x + a_6$$

where a_4 is equal to

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Using the method of Faltings and Serre, we verify that J is indeed isogeneous to A.

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Algorithmic methods

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$$S_2(N) \cong H_1(X_0(N), \mathbb{C}; \mathrm{cusps})^+$$

where the $+$ indicates the $+$-space for complex conjugation.

When $B \not\cong \mathrm{M}_2(\mathbb{Q})$ (no cusps), we can still identify

$$S_2^B(\mathfrak{N}) \cong H^1(X_0(\mathfrak{N}), \mathbb{C})^+ \cong H^1(\Gamma_0(\mathfrak{N}), \mathbb{C})^+ \cong \mathrm{Hom}(\Gamma_0(\mathfrak{N}), \mathbb{C})^+;$$

we compute this space as a Hecke module by working explicitly with a presentation for the group $\Gamma_0(\mathfrak{N})$, using an algorithm for quaternionic ideal principalization for the Hecke operators.

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So the case left is when $n = [F : \mathbb{Q}]$ is odd and F with (strict) class number > 1; in this case, the natural object with an action of Hecke operators is

$$X(\mathbb{C}) = \bigsqcup_{[\mathfrak{b}] \in \mathrm{Cl}^+ \mathbb{Z}_F} X_{\mathfrak{b}}(1)(\mathbb{C}),$$

a disjoint union of curves indexed by the strict class group of F.

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$$i, j \mapsto \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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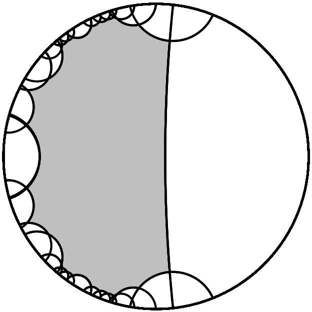

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We write $\delta'_{[1:0]}$ as a word in the generators for Γ', repeat, and sum.

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H^+ breaks up, yielding two one-dimensional eigenforms f and g.

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$w+3$	5	1	1
2	8	-5	-5
$2w+7$	9	-2	2
w	11	0	0
w^2-w-8	17	-5	5
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The form g is visibly a quadratic twist of f.

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and in particular, J miraculously has an 11-isogeny.

Thanks!