Huff's Model for Elliptic Curves

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Outline

- Elliptic curves and elliptic curves models
- Huff's model
- Efficient arithmetic on Huff curves
- Generalizations and extensions
- Efficient pairings on Huff curves

Definition (Elliptic curve)

A nonsingular absolutely irreducible projective curve defined over a field $\mathbb{F}$ of genus 1 with one distinguished $\mathbb{F}$-rational point is called an elliptic curve over $\mathbb{F}$

- An elliptic curve E over $\mathbb{F}$ can be given by the so-called Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

where the coefficients $a_1, a_2, a_3, a_4, a_6 \in \mathbb{F}$

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- The set of $\mathbb{F}$-rational points on E is defined by the set of points

  $$E(\mathbb{F}) = \{(x, y) \in \mathbb{F}\times\mathbb{F} : y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6\}\cup\{P_\infty\}$$

  where P_∞ is the point at infinity
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Efficient Arithmetic

- Finite field arithmetic
- Elliptic curve arithmetic
  - The shape of the curve
  - The coordinate systems
    - Addition formulas: What is the cost? Is it unified? Is it complete?
  - Scalar multiplication
- Evaluation of pairings

Some Forms of Elliptic Curves

- There are many ways to represent an elliptic curve such as

Long Weierstrass:	$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$
Short Weierstrass:	$y^2 = x^3 + ax + b$
Legendre:	$y^2 = x(x-1)(x-\lambda)$
Montgomery:	$by^2 = x^3 + ax^2 + x$
Doche-Icart-Kohel:	$y^2 = x^3 + 3a(x+1)^2$
Jacobi intersection:	$x^2 + y^2 = 1, ax^2 + z^2 = 1$
Jacobi quartic:	$y^2 = x^4 + 2ax^2 + 1$
Hessian:	$x^3 + y^3 + 1 = 3dxy$
Edwards:	$x^2 + y^2 = c^2(1 + x^2y^2)$
Twisted Edwards:	$ax^2 + y^2 = 1 + dx^2y^2$

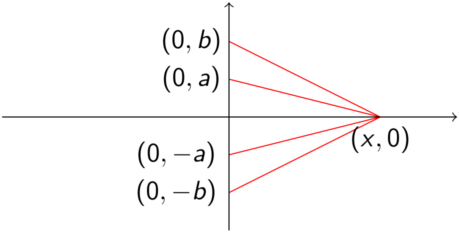
- Some of these define curves with singular projective closures but geometric genus 1

- There are several ways to represent an elliptic curve over a field of characteristic 2 such as

$$\begin{aligned} \text{Long Weierstrass:}\quad & y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \\ \text{Short Weierstrass:}\quad & y^2 + xy = x^3 + ax^2 + b \\ \text{Hessian:}\quad & x^3 + y^3 + 1 = dxy \\ \text{Binary Edwards:}\quad & c(x+y) + d(x^2+y^2) = xy + xy(x+y) + x^2y^2 \end{aligned}$$

A Diophantine problem

$a, b \in \mathbb{Q}^*, \ a^2 \neq b^2$



$\boxed{x \in \mathbb{Q} \text{ for which } (x, 0) \text{ is at rational distances from } (0, \pm a) \text{ and } (0, \pm b) \text{ ?}}$

equivalent to

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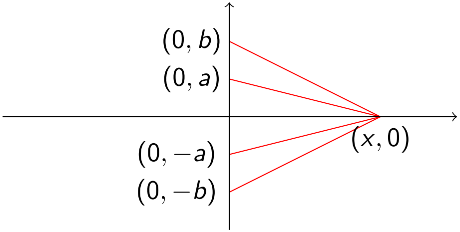

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Huff's Model

Gerald B. Huff. Diophantine problems in geometry and elliptic ternary forms. Duke Math. J., 15:443–453, 1948.

$$aX(Y^2 - Z^2) = bY(X^2 - Z^2)$$

- defines an elliptic curve if $a^2 \neq b^2$ and $a, b \neq 0$ over any field $\mathbb{K}$ of odd characteristic with $(0 : 0 : 1)$ as the neutral element,
- with three points at infinity $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(a : b : 0)$
- isomorphic to the Weierstrass form:

$$V^2 W = U(U + a^2 W)(U + b^2 W)$$

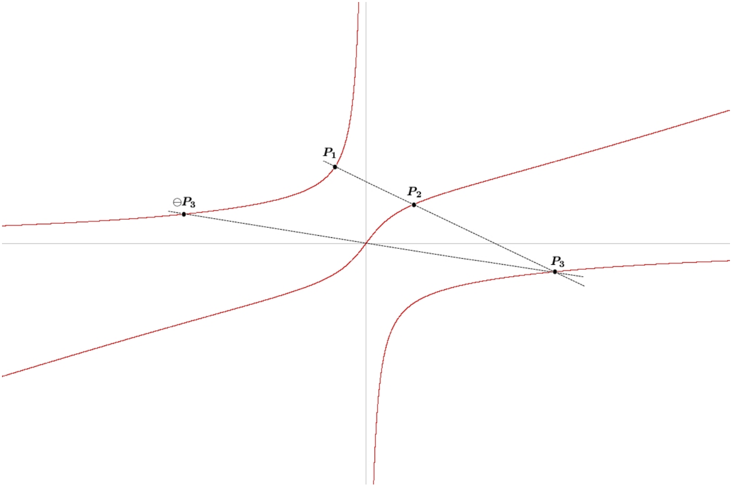
(with $(U : V : W) = \big(ab(bX - aY) : ab(b^2 - a^2)Z : -aX + bY\big)$)

Huff's Model

$$E : aX(Y^2 - Z^2) = bY(X^2 - Z^2)$$

- $\mathbf{O} = (0 : 0 : 1)$ is an inflection point of $E \rightsquigarrow (E, \mathbf{O})$ is an elliptic curve with $\mathbf{O}$ as neutral element
- chord-and-tangent group law on E
- $\rightsquigarrow$ the inverse of $\mathbf{P_1} = (X_1 : Y_1 : Z_1)$ is $\ominus\mathbf{P_1} = (X_1 : Y_1 : -Z_1)$
- $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(a : b : 0)$ are 2-torsion points of E $(\pm 1 : \pm 1 : 1)$ are 4-torsion points; these points form a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- conversely, in odd characteristic, any elliptic curve with a rational subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is isomorphic to a Huff curve (Riemann-Roch exercise)

Huff's Model



$$ax(y^2 - 1) = by(x^2 - 1)$$

Unified/Complete Addition Formulas

$$E : ax(y^2 - 1) = by(x^2 - 1), \qquad \mathbf{P_1} \oplus \mathbf{P_2} \oplus \mathbf{P_3} = \mathbf{O}$$

- $\mathbf{P_1} = (x_1, y_1)$, $\mathbf{P_3} = (x_2, y_2)$, $\mathbf{P_3} = (-x_3, -y_3)$ with

$$x_3 = \frac{(x_1 + x_2)(1 + y_1 y_2)}{(1 + x_1 x_2)(1 - y_1 y_2)} \text{ and } y_3 = \frac{(y_1 + y_2)(1 + x_1 x_2)}{(1 - x_1 x_2)(1 + y_1 y_2)}$$

  whenever $x_1 x_2 \neq \pm 1$ and $y_1 y_2 \neq \pm 1$

- addition law is *unified*: it can be used to double a point
- involves inversions $\rightsquigarrow$ projective coordinates:

$$\begin{cases} X_3 = (X_1 Z_2 + X_2 Z_1)(Y_1 Y_2 + Z_1 Z_2)^2 (Z_1 Z_2 - X_1 X_2) \\ Y_3 = (Y_1 Z_2 + Y_2 Z_1)(X_1 X_2 + Z_1 Z_2)^2 (Z_1 Z_2 - Y_1 Y_2) \\ Z_3 = ({Z_1}^2 {Z_2}^2 - {X_1}^2 {X_2}^2)({Z_1}^2 {Z_2}^2 - {Y_1}^2 {Y_2}^2) \end{cases}$$

  can be evaluated with 12m

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Applicability

- The previous addition formula on a Huff curve is *independent* of the curve parameters

- Moreover, it is almost complete:

Theorem

Let $\mathbf{P_1} = (X_1 : Y_1 : Z_1)$ *and* $\mathbf{P_2} = (X_2 : Y_2 : Z_2)$ *be two points on a Huff curve. Then the previous addition formula is valid provided that* $X_1X_2 \neq \pm Z_1Z_2$ *and* $Y_1Y_2 \neq \pm Z_1Z_2$.

- in particular, if $\mathbf{P}$ is of odd order, the addition law in $\langle \mathbf{P} \rangle$ is complete
- useful $\rightsquigarrow$ natural protection against certain *side-channel attacks*

- The doubling formula can be sped up by evaluating squarings
  The cost of a point doubling then becomes $7\mathbf{m} + 5\mathbf{s}$ or $10\mathbf{m} + 1\mathbf{s}$

- Choosing $\mathbf{O}' = (0:1:0)$ as the neutral element results in translating the group law. We get

$$\begin{cases} X_3 = (X_1Z_2 + X_2Z_1)(Y_1Y_2 + Z_1Z_2)(Y_1Z_2 + Y_2Z_1) \\ Y_3 = (X_1X_2 - Z_1Z_2)({Z_1}^2{Z_2}^2 - {Y_1}^2{Y_2}^2) \\ Z_3 = (Y_1Z_2 + Y_2Z_1)(X_1X_2 + Z_1Z_2)(Y_1Y_2 - Z_1Z_2) \end{cases}$$

  This *unified* addition formula can be evaluated with $11\mathbf{m}$
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- **Twisted curves**: Let $P \in \mathbb{K}[T]$ be a monic polynomial of degree 2, with non-zero discriminant, and such that $P(0) \neq 0$. We can generalize Huff's model and introduce the cubic curve

$$axP(y) = byP(x) \text{ where } a, b \in \mathbb{K}^*$$

With $P(T) = T^2 - d$, the sum of two points can be evaluated with 12m using projective coordinates

- **Binary fields**: Huff's form can be extended to a binary field as

$$ax(y^2 + y + 1) = by(x^2 + x + 1)$$

with neutral element $\mathbf{O} = (0, 0)$

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Pairings

- E pairing-friendly elliptic curve over $\mathbb{F}_q$ with hn rational points (n a large prime) and even embedding degree k wrt n (i.e. $n | q^k - 1$)
- To compute e.g. the (reduced) Tate pairing:

$$T_n \colon E(\mathbb{F}_q)[n] \times E(\mathbb{F}_{q^k})/[n]E(\mathbb{F}_{q^k}) \longrightarrow \mu_n$$

one typically uses Miller's algorithm

Algorithm 1 Miller's algorithm for T_n, $n = \overline{n_{\ell-1} n_{\ell-1} \cdots n_0}^2$

```
f ← 1; R ← P
for i = ℓ − 2 down to 0 do
    f ← f^2 · g_{R,R}(Q); R ← [2]R          ▷ Miller doubling
    if (n_i = 1) then
        f ← f · g_{R,P}(Q); R ← R ⊕ P       ▷ Miller addition
    end if
end for
return f^{(q^k − 1)/n}
```

Pairings on Huff curves

- The rational function $g_{\mathbf{R},\mathbf{P}}$ in Miller's algorithm is Miller's line function, with divisor $\mathbf{R} + \mathbf{P} - \mathbf{O} - (\mathbf{R} \oplus \mathbf{P})$
- The faster we can compute $g_{\mathbf{R},\mathbf{P}}$, the faster the pairing
- For a curve with chord-and-tangent addition, like Weierstrass or Huff but unlike Edwards, $g_{\mathbf{R},\mathbf{P}}$ is simple: equation of the line through $\mathbf{R}$ and $\mathbf{P}$
- Huff curves have a simple line function and efficient arithmetic $\rightsquigarrow$ convenient for pairings?

- The formulas we find don't use the fastest possible addition or doubling $\rightsquigarrow$ not far from the records set for Jacobian coordinates or Edwards, but not as fast
- Actual multiplication counts:

$$\begin{aligned} \text{mixed Miller addition:}\quad & 1\mathrm{M} + (k+13)\mathrm{m} \\ \text{full Miller addition:}\quad & 1\mathrm{M} + (k+15)\mathrm{m} \\ \text{Miller doubling:}\quad & 1\mathrm{M} + 1\mathrm{S} + (k+11)\mathrm{m} + 6\mathrm{s} \end{aligned}$$

- Compares to Arène et al.'s records: $1\mathrm{M} + (k+12)\mathrm{m}$, $1\mathrm{M} + (k+14)\mathrm{m}$, $1\mathrm{M} + 1\mathrm{S} + (k+6)\mathrm{m} + 5\mathrm{s}$ for Edwards
- Room for improvement!

Summary

- Altenate representation for elliptic curves
- Efficient arithmetic
- Useful properties
  - unified/complete addition law
  - addition law independent of curve parameters
- Suitable for pairing evaluation

Comments/Questions?