On the extremality of an 80-dimensional lattice

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The result

THEOREM

One of the even unimodular lattices associated to the length 80 extended (binary) quadratic residue code is extremal:
the minimal non-zero norm is 8. We have $\mathbf{SL}_2(\mathbf{F}_{79}) \subseteq \mathbf{Aut}(L)$.

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- L and its link to coding theory (via cyclotomy) was codified by Schulze-Pillot, who could not find any norm 6 vector.
- If $n > 80$, a lattice related to QR codes cannot be extremal. (sqrt bound on minimum versus linear growth requirement).
- No extremal lattice known for $n = 72$ (or $n > 80$).
- Bachoc and Nebe previously found two other extremal lattices with $n = 80$, via quaternionic coding theory.
- The known part of our $\mathbf{Aut}(L)$ is smaller: $8.3 \cdot 10^6 > 4.9 \cdot 10^5$.

The techniques

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- We show no norm 6 by finding **all** norm 10 vectors (!).
- This is valid, using the Θ-series positivity.
- In the short lattice vector enumeration, we use tree pruning.
- We also use nice **Aut** action (as doubly transitive signed permutations), derived in part by Abel, Elkies and Kominers.

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The enumeration part is heuristic, but we still get a proved result.

Plan

1- **Reminders.**
2- Overview of the strategy.
3- Lattice enumeration.

Lattices

Lattice $\equiv$ additive subgroup of $\mathbb{Z}^n$

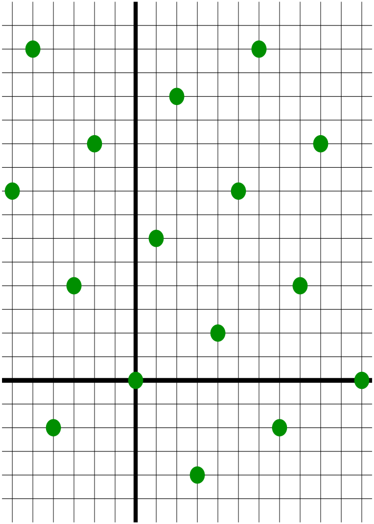
$\equiv \quad \{\sum_{i \leq n} x_i \mathbf{b}_i : x_i \in \mathbb{Z}\}$

First minimum:
$\lambda = \min(\|\mathbf{b}\|^2 : \mathbf{b} \in L \setminus \mathbf{0})$.

Lattice volume:
$\det L = |\det(\mathbf{b}_i)_i|$, for any basis.

Unimodular lattice: $|\det L| = 1$.
Even lattice: $\|\mathbf{b}\|^2$ even for all $\mathbf{b} \in L$.

Famous even unimod. lattices: E_8, L_{24}.



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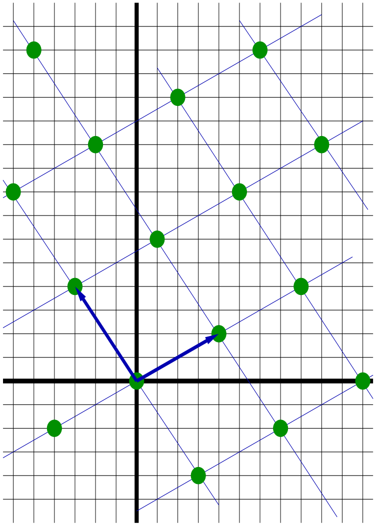

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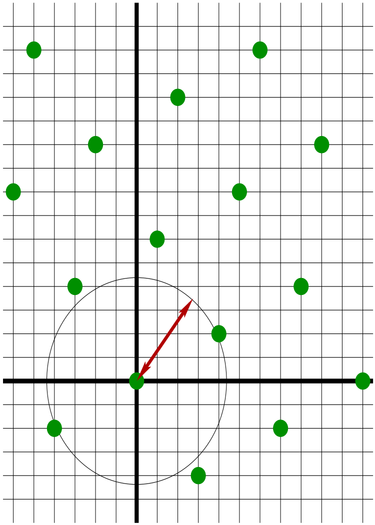

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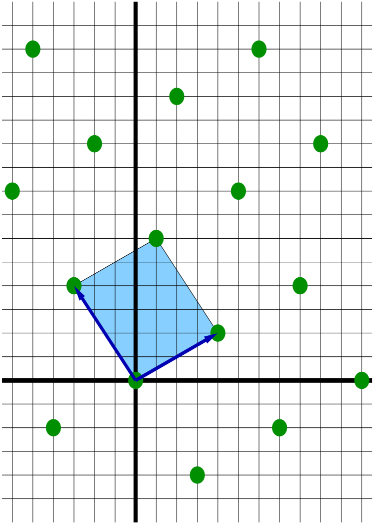

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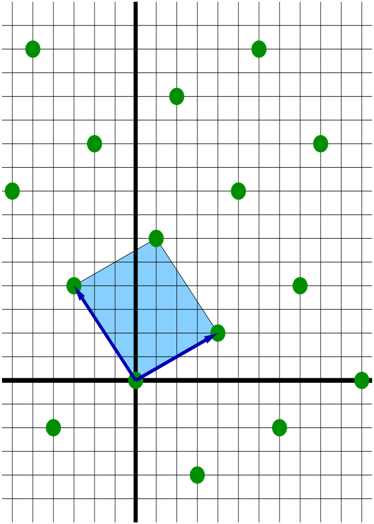

Theta series

- Theta-series: $\Theta(L) = \sum_{\mathbf{b} \in L} q^{\|\mathbf{b}\|^2/2}$ (non-negative coeffs).

If L is an even unimodular lattice L of dimension 8ℓ,
then $\Theta(L)$ is a modular form of weight 4ℓ.

- The set of modular forms of weight 4ℓ is a vector space of dimension $d = 1 + \lfloor 8\ell/24 \rfloor$.
- A triangular basis for this vector space looks like

$$\begin{array}{lclcccc} f_0 & = & 1+ & & & c_{d,0}\, q^d & + \ldots \\ f_1 & = & & q+ & & c_{d,1}\, q^d & + \ldots \\ & & & \ldots & & & \\ f_{d-1} & = & & & q^{d-1}+ & c_{d,d-1}\, q^d & + \ldots \end{array}$$

- An even unimodular L is said **extremal** if $\Theta(L) = f_0$.

General comments about extremality

- For large enough n, f_0 has negative coeffs.
  $\Rightarrow$ The total number of extremal lattices is bounded:

$$n \leq 163\,264 \quad \& \quad \text{genus theory in fixed } n.$$

- If n not a multiple of 8, minus signs abound, so no extremality is possible (for our definition).

Number of known extremal lattices:

8	16	24	32	40	48	56	64	72	80
1	2	1	$\geq 10^7$	$\geq 10^{51}$	3	3	1	0	2(+1)
E_8	$E_8 \oplus E_8$, D_{16}^+	L_{24}	mass formula						

Plan

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2- **Overview of the strategy.**

3- Lattice enumeration.

Plan to show extremality

Case of 80-dimensional lattices (weight 40 modular forms):

$$
\begin{aligned}
f_0 &= 1 + 1250172000\, q^4 + 7541401190400\, q^5 + O(q^6)\\
f_1 &= q + 19291168\, q^4 + 37956369150\, q^5 + O(q^6)\\
f_2 &= q^2 + 156024\, q^4 + 57085952\, q^5 + O(q^6)\\
f_3 &= q^3 + 168\, q^4 - 12636\, q^5 + O(q^6)
\end{aligned}
$$

- We have $\Theta(L) = f_0 + a_1 f_1 + a_2 f_2 + a_3 f_3$ for integers $a_i \geq 0$.
- L has no vector of norm ≤ 4 (via a coding theory analogy):

$$\Rightarrow \quad a_1 = a_2 = 0.$$

- Find $\approx 7.5 \cdot 10^{12}$ vectors of norm 10.
  Positivity gives $a_3 = 0$, due to the minus sign on "$12636\, q^5$".
- We use heuristics & automorphisms to get norm 10 vectors.

Searching vectors of norm 10 rather than norm 6???

- We would need to **provably exhaust** all norm 6 vectors.
- We **heuristically** find a **tiny** subset of the norm 10 vectors.
- We estimate the speed-up to be around 1000.

Principle: Apply **Aut**(L) to reduce search space.

Remark: This strategy could be used for $n = 72$ (with $10 \to 8$) and for $n = 88$ (with $10 \to 12$).

Construction of our lattice L

- The construction of L and the methods used to accelerate the finding of short vectors are independent.
- But finding a canonical representative of the orbit class of a vector (under **Aut**) requires some knowledge of the group action on L.
- Elkies modified a construction of Gross to get five 80-dim lattices, in correspondence with the class group of $\mathbf{Q}(\sqrt{-79})$. Each can be given in a basis s.t.
  - All coords have the same parity,
  - The square-sum of the coords is 16x the vector norm.
- This yields the same lattices as Schulze-Pillot's, only one of which is a candidate for extremality: L.

Apply **Aut**(L) to reduce search space

- The (known) automorphisms have a 'nice' action on Elkies' basis: doubly transitive signed permutations on coords.
  $\Rightarrow$ Finding canonical representatives of orbit classes is easy.
- Finding $\approx 7.5 \cdot 10^{12}$ vectors of norm 10 is reduced by a factor
  $$\sim \#\mathbf{SL}_2(\mathbf{F}_{79}) \approx 4.9 \cdot 10^5.$$

We first eliminate vectors with non-trivial stabilisers:

- Take $g \in \mathbf{Aut}(L)$ of nontrivial conjugacy class, and find all short vectors in lattices $\mathrm{Ker}(g - I)$ (dim ≤ 28).
- We are left to find $N \approx 1.5 \cdot 10^7$ norm 10 orbits.
- Via coupon-collecting analysis, we expect to need $\sum_{k \leq N} \frac{N}{k} \approx 2.5 \cdot 10^8$ "random" norm 10 vectors.

Plan

1- Reminders.

2- Overview of the strategy.

3- **Lattice enumeration.**

The Kannan-Fincke-Pohst algorithm

Let $(\mathbf{b}_i)$ be a basis of L. Goal: $\|\sum_i x_i \mathbf{b}_i\|^2 \leq 10$ with $x_i \in \mathbb{Z}$.

- Gram-Schmidt orthogonalisation: $\mathbf{b}_i^\star = \mathbf{b}_i - \sum_{j<i} \mu_{i,j} \mathbf{b}_j^\star$.
- Shifts: $y_i := x_i + \sum_{j>i} \mu_{j,i} x_j$.

$\Rightarrow$ New goal: $\sum_i y_i^2 \|\mathbf{b}_i^\star\|^2 \leq 10$.

KFP algorithm:

- Try all y_d s.t. $y_d^2 \|\mathbf{b}_d^\star\|^2 \leq 10$.
- Try all (y_{d-1}, y_d) s.t. $\sum_{i \geq d-1} y_i^2 \|\mathbf{b}_i^\star\|^2 \leq 10$.

$\vdots$

- Try all $(y_2, \ldots, y_d)$ s.t. $\sum_{i \geq 2} y_i^2 \|\mathbf{b}_i^\star\|^2 \leq 10$.
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Pruning the KFP tree

Principle: Don't waste all the norm on large i!

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Used $P_j = 1 - \frac{j-1}{100}$, which seemed good in practice.

MAGMA traverses this KFP tree at $\approx$ 7.5 million nodes/second.

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Pruning the KFP tree

Principle: Don't waste all the norm on large i!

Pruned KFP algorithm:

- Try all y_d s.t. $y_d^2\|\mathbf{b}_d^\star\|^2 \leq P_d \cdot 10.$
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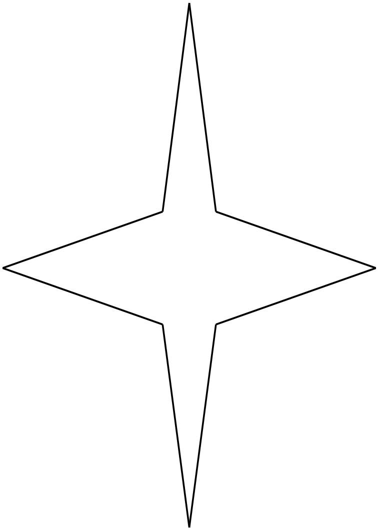
Refreshing the basis



- Schnorr-Euchner tree traversal.
- Random basis change every 10^5 vecs (30 mins) $\Rightarrow$ trivial parallelisation.
- $\sim$300,000 nodes per vector found.
- Can heuristically analyze the miss rate and subtrees sizes via volumes of truncated hyperspheres.
- Resembles the "extreme pruning" from [Gama et al, Eurocrypt'10].

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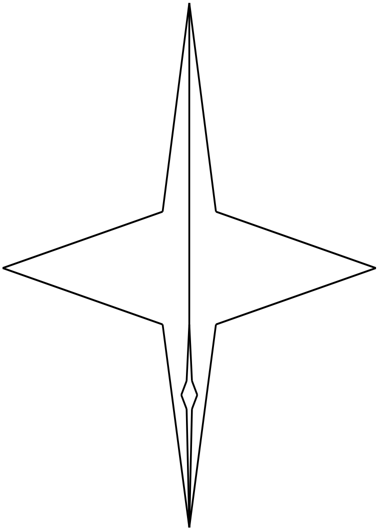

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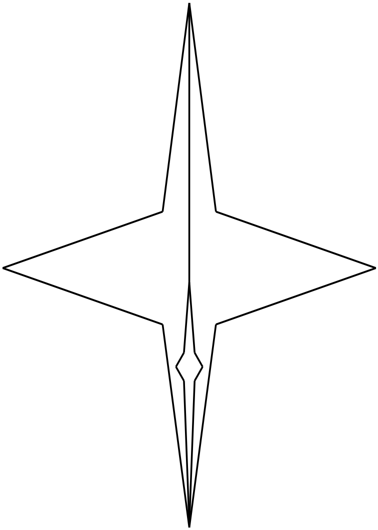

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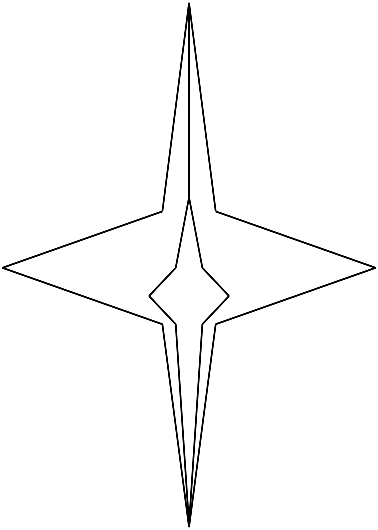

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Concluding remarks

- Our code (in Magma/C) ran in 4 days using 14 CPUs.
- The data can be checked in about 10 hours on 1 CPU.
- ≈90% time in finding vectors, 5% canonical orbit reps.

- There are at least 3 other "candidates" for $n = 80$, though the **Aut** groups are not as nice.
- No extremal candidate is known (to us) for $n = 72$.
- We can prove that L is not isometric to the Bachoc-Nebe lattices, using the Classification of Finite Simple Groups.

- For more details, read the paper. ☺