Lattices and spherical designs.

Gabriele Nebe

Lehrstuhl D für Mathematik

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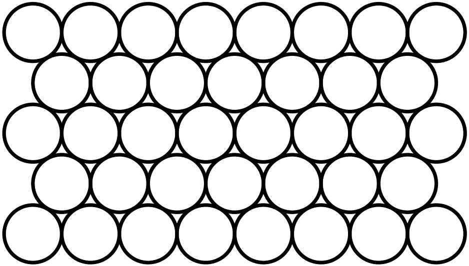
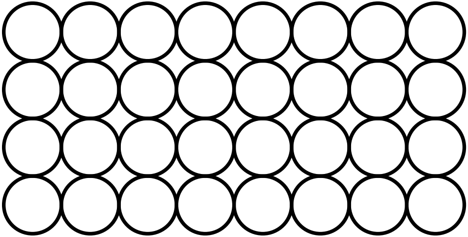
Lattice sphere packings.





Lattices.

- $B = (B_1, \ldots, B_n)$ basis of Euclidean space $(\mathbb{R}^n, (\,,\,))$.
- $L = \{\sum_{i=1}^n a_i B_i \mid a_i \in \mathbb{Z}\}$ lattice.
- $\min(L) := \min\{(\ell, \ell) \mid 0 \neq \ell \in L\}$ minimum of L.
- For $a := \sqrt{\min(L)}/2$ the associated lattice sphere packing is $\mathcal{P}(L) := \dot{\bigcup}_{\ell \in L} B_a(\ell)$.
- Main goal in lattice theory:
  Find dense lattices.
  Classify all densest lattices in a given dimension.
  Classify densest lattices in certain families of lattices.

Theorem.

The densest lattices are known up to dimension 8 and in dimension 24.

n	1	2	3	4	5	6	7	8	24
L	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{D}_4$	$\mathbb{D}_5$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	Λ_{24}
extreme	1	1	1	2	3	6	30	2408	

Voronoi's characterization.

- The space of similarity classes of n-dimensional lattices is a compact Riemannian manifold.
- There are only finitely many similarity classes of locally densest lattices: extreme lattice ($n = 8$, 2408 extreme lattices)
- Voronoi gave a characterization of extreme lattices by the geometry of the minimal vectors $\mathrm{Min}(L) := \{\ell \in L \mid (\ell, \ell) = \min(L)\}$.
- L is perfect if $\{\pi_x := x^{tr} x \mid x \in \mathrm{Min}(L)\} = \mathbb{R}^{n \times n}_{sym}$.
- L is eutactic if there are $\lambda_x > 0$ such that $I_n = \sum_{x \in \mathrm{Min}(L)} \lambda_x \pi_x$.
- L is strongly eutactic if all λ_x can be chosen to be equal.

Theorem (Voronoi, 1908)

L is extreme, if and only if L is perfect and eutactic.

Strongly perfect lattices.

Definition (B. Venkov)

A lattice L is called strongly perfect if $\mathrm{Min}(L)$ is a spherical 5-design, so if for all $p \in \mathbb{R}[x_1, \ldots, x_n]_{deg \leq 5}$

$$\frac{1}{|\mathrm{Min}(L)|} \sum_{x \in \mathrm{Min}(L)} p(x) = \int_S p(t)dt$$

where S is the sphere containing $\mathrm{Min}(L)$.

Equivalent are the following.

- $X := \mathrm{Min}(L)$ is a 5-design.
- $X := \mathrm{Min}(L)$ is a 4-design.
- $\sum_{x \in X} f(x) = 0$ for all harmonic polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree 2 and 4.
  (harmonic means homogeneous and $\Delta(f) = \sum \frac{d^2 f}{dx_i^2} = 0$).

Continued.

Equivalent are the following.

- $X := \mathrm{Min}(L)$ is a 5-design.
- $X := \mathrm{Min}(L)$ is a 4-design.
- $\sum_{x\in X} f(x) = 0$ for all harmonic polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree 2 and 4.
- There is some $c \in \mathbb{R}$ such that $\sum_{x\in X}(x,\alpha)^4 = c(\alpha,\alpha)^2$ for all $\alpha \in \mathbb{R}^n$.
- $$\begin{array}{lll} (D4) & \sum_{x\in X}(x,\alpha)^4 & = \frac{3|X|m^2}{n(n+2)}(\alpha,\alpha)^2 \\ (D2) & \sum_{x\in X}(x,\alpha)^2 & = \frac{|X|m}{n}(\alpha,\alpha) \end{array}$$

  for all $\alpha \in \mathbb{R}^n$ where $m = \min(L)$.

Strongly perfect lattices are extreme.

Theorem.

Let L be a strongly perfect lattice. Then L is strongly eutactic and perfect and hence extreme.

Proof. (a) The 2-design property is equivalent to L being strongly eutactic, because by (D2)

$$\sum_{x \in X} \underbrace{(x, \alpha)^2}_{\alpha \pi_x \alpha^{tr}} = \frac{m|X|}{n} \underbrace{(\alpha, \alpha)}_{\alpha I_n \alpha^{tr}}$$

for all $\alpha \in \mathbb{R}^n$ where $X = \mathrm{Min}(L)$, $m = \min(L)$.

Strongly perfect lattices are extreme.

Theorem.

Let L be a strongly perfect lattice. Then L is strongly eutactic and perfect and hence extreme.

Proof. (b) 4-design implies perfection: $A \in \mathbb{R}^{n\times n}_{sym}$ defines $p_A : \alpha \mapsto \alpha A \alpha^{tr}$.

$$U := \langle \pi_x \mid x \in X \rangle = \mathbb{R}^{n\times n}_{\mathrm{sym}} \Leftrightarrow U^{\perp} = \{0\}.$$

So assume that $A \in U^{\perp}$, so

$$0 = \mathrm{trace}(x^{tr} x A) = \mathrm{trace}(x A x^{tr}) = x A x^{tr} = p_A(x) \text{ for all } x \in X$$

By the design property we then have

$$\int_S p_A^2(t)dt = \frac{1}{|X|} \sum_{x\in X} p_A(x)^2 = 0$$

and hence $A = 0$.

Strongly perfect lattices.

Theorem.

Let L be strongly perfect. Then $\min(L)\min(L^{\#}) \geq (n+2)/3$. Here $L^{\#} = \{x \in \mathbb{R}^n \mid (x,L) \subset \mathbb{Z}\}$ is the dual lattice.

Proof. Let $\alpha \in \mathrm{Min}(L^{\#})$. Then

$$(D4)-(D2) = \sum_{x\in X} \underbrace{(x,\alpha)^2((x,\alpha)^2-1)}_{\geq 0} = \frac{|X|m}{n}(\alpha,\alpha)\underbrace{\left(\frac{3m(\alpha,\alpha)}{n+2}-1\right)}_{\Rightarrow \geq 0}$$

Remember

$$\begin{array}{lll} (D4) & \sum_{x\in X}(x,\alpha)^4 & = \frac{3|X|m^2}{n(n+2)}(\alpha,\alpha)^2 \\ (D2) & \sum_{x\in X}(x,\alpha)^2 & = \frac{|X|m}{n}(\alpha,\alpha) \end{array}$$

Dual strongly perfect lattices.

Definition

Let L be a lattice and $L^{\#}$ its dual lattice.

- For $a \in \mathbb{R}_{\geq 0}$ the layer $L_a := \{\ell \in L \mid (\ell, \ell) = a\}$ is a finite subset of a sphere.
- L is called universally strongly perfect if all layers of L form spherical 4-designs.
- L is called dual strongly perfect if L and $L^{\#}$ are both strongly perfect.

Theorem.

universally strongly perfect $\Rightarrow$ dual strongly perfect $\Rightarrow$ strongly perfect

Proof. Theta series of L $\quad \theta_L := \sum_a |L_a| q^a \quad (q = \exp(\pi i z), \Im(z) > 0)$
or more general $\theta_{L,p} := \sum_a \sum_{x \in L_a} p(x) q^a$ for $p \in Harm_d$ are modular forms.
L universally strongly perfect, iff $\theta_{L,p} = 0$ for all $p \in Harm_d$ $(d = 2, 4)$.
$\theta_{L^{\#},p}$ can be computed from $\theta_{L,p}$ by Poisson-summation.

No harmonic invariants.

Theorem.

Let $G = \mathrm{Aut}(L)$ and assume that $\langle (\alpha, \alpha)^d \rangle = \mathrm{Inv}_{2d}(G)$ for all $d = 1, \ldots, t$. Then all G-orbits and all non-empty layers of L are spherical $2t$-designs.

Corollary.

- If $\mathbb{R}^n$ is an irreducible $\mathbb{R}G$-module then $\mathrm{Inv}_2(G) = \langle (\alpha, \alpha) \rangle$ and L is strongly eutactic.
- In particular all irreducible root-lattices are strongly eutactic.
- If additionally $\mathrm{Inv}_4(G) = \langle (\alpha, \alpha)^2 \rangle$, then L is universally strongly perfect.

The Thompson-Smith lattice of dimension 248.

- Let $G =$Th denote the sporadic simple Thompson group.
- Then G has a 248-dimensional rational representation $\rho : G \to O(248, \mathbb{Q})$.
- Since G is finite, $\rho(G)$ fixes a lattice $L \leq \mathbb{Q}^{248}$.
- Modular representation theory tells us that for all primes p the $\mathbb{F}_pG$-module L/pL is simple.
- Therefore $L = L^{\#}$ and L is even
- otherwise $L_0 := \{v \in L \mid (v,v) \in 2\mathbb{Z}\} < L$ of index 2.
- $\mathrm{Inv}_{2d}(G) = \langle (\alpha,\alpha)^d \rangle$ for $d = 1,2,3$. So all layers of L form spherical 6-designs and in particular L is strongly perfect.
- $\min(L)\min(L^{\#}) = \min(L)^2 \geq \frac{248+2}{3} > 83.3$, so $\min(L) \geq 10$.
- There is a $v \in L$ with $(v,v) = 12$, so $\min(L) \in \{10, 12\}$.

Classification of strongly perfect lattices.

Theorem.

- All strongly perfect lattices of dimension ≤ 12 are known (Nebe/Venkov).
- All integral strongly perfect lattices of minimum 2 and 3 are known (Venkov).
- There is a unique dual strongly perfect lattice of dimension 14 (Nebe/Venkov).
- Elisabeth Nossek classifies the dual strongly perfect lattices in dimension 13,15,$\ldots$ in her thesis.
- All integral lattices L of minimum ≤ 5 such that $\mathrm{Min}(L)$ is a 6-design are known (Martinet).
- All lattices L of dimension ≤ 24 such that $\mathrm{Min}(L)$ is a 6-design are known (Nebe/Venkov).

Extremal lattices are extreme.

Theorem.

Let L be an even unimodular lattice of dimension $n = 24a + 8b$ with $b = 0, 1, 2$ and $\min(L) = 2a + 2$ (extremal lattice).

- All nonempty L_j are $(11 - 4b)$-designs.
- If $b = 0$ or $b = 1$ then L is strongly perfect and hence extreme.
- All extremal even unimodular lattices of dimension 32 are extreme.

Proof:

- Let $L = L^{\#} \subset \mathbb{R}^n$ be an even unimodular lattice.
- Choose $p \in \mathbb{R}[x_1, \ldots, x_n]$, $\deg(p) = t > 0$, $\Delta(p) = 0$.
- Then $\theta_{L,p} := \sum_{\ell \in L} p(\ell) q^{(\ell,\ell)} = \sum_{j=1}^{\infty} (\sum_{\ell \in L_j} p(\ell)) q^j$ is a cusp form of weight $n/2 + t$.
- If $2m = \min(L)$ then $\theta_{L,p}$ is divisible by Δ^m of weight $12m$
- If $n/2 + t < 12m$, then $\theta_{L,p} = 0$ and all layers of L are spherical t-designs.

Strongly perfect lattices: Conclusion.

- Boris Venkov's idea combines spherical designs and lattices
- Allows to apply other mathematical theories to prove that certain lattices are locally densest such as:
- Representation theory of finite groups.
- Theory of modular forms.
- Combinatorics:
- Explicit knowledge of minimal vectors (Barnes-Wall lattices)
- Allows to use combinatorial means to classify strongly perfect lattices of given dimension.
- Classification of dual strongly perfect lattices: Many more tools. (Finite list of abelian groups $L^{\#}/L$, finite list of possible genera of lattices, use modular forms or explicit enumeration of genera.)

Spherical designs.

Definition

A finite set $\emptyset \neq X \subset S := S^{n-1}(\mathbb{R}) := \{x \in \mathbb{R}^n \mid (x,x) = 1\}$ is called spherical t-design if for all $p \in \mathbb{R}[x_1, \ldots, x_n]_{\leq t}$

$$\frac{1}{|X|} \sum_{x \in X} p(x) = \int_S p(t) dt.$$

Clear: X is a t-design $\Rightarrow$ X is a $t-1$-design.
Disjoint unions of t-designs are t-designs.
Fact: t designs exist for arbitrary t and n.

Goal.

Find designs of minimal cardinality, so called tight designs.

$$|X| \geq \binom{n+e-1}{e} + \binom{n+e-2}{e-1} \text{ resp. } 2\binom{n+e-1}{e}$$

for $t = 2e$ resp. $t = 2e+1$.

Classification of tight spherical t-designs.

Remark

Tight t-designs in S^{n-1} with $n \geq 3$ only exist for $t \leq 5$ or $t = 7, 11$.
They are classified completely for $t \in \{1, 2, 3, 11\}$.

Examples

- $n = 2$: regular (t+1)-gon
- $t = 1$: $|X| = 2\binom{n-1}{0} = 2$, $X = \{x, -x\}$
- $t = 2$: $|X| = n + 1$, simplex.
- $t = 3$: $|X| = 2\binom{n}{1} = 2n$, $X = \{\pm e_1, \ldots, \pm e_n\} = \mathrm{Min}(\mathbb{Z}^n)$.
- $t = 5$: $n = 3$, $|X| = 12$, icosahedron.
- $t = 7$: $n = 8$ and $X = \mathrm{Min}(\mathbb{E}_8)$, $|X| = 240$.
- $t = 7$: $n = 23$ and $X = \mathrm{Min}(O_{23})$, $|X| = 4600$.
- $t = 11$: $n = 24$ and $X = \mathrm{Min}(\Lambda_{24})$, $|X| = 196560$. unique.

Tight spherical designs.

Tight spherical designs, known facts.

- Only exist for $n \leq 2$ or $t = 1, 2, 3, 4, 5, 7, 11$.
- Classified for $n \leq 2$ or $t = 1, 2, 3, 11$.
- Open for $t = 4, 5, 7$.
- $\{Y \subset S^{n-1} \mid Y$ tight 5-design$\} \leftrightarrow \{X \subset S^{n-2} \mid X$ tight 4-design$\}$
- t odd $\Rightarrow$ any tight t-design is antipodal: $X = -X$.
- $t = 4$, $|X| = n(n+3)/2$, then either $n = 2$ or $n = (2m+1)^2 - 3 =$ 6, 22, but not 46, 78, open for $n \geq 118$.
- $t = 5$, $|X| = n(n+1)$, then either $n = 3$ or $n = (2m+1)^2 - 2 =$ 7, 23, but not 47, 79, open for $n \geq 119$.
- $t = 7$, $|X| = n(n+1)(n+2)/3$, then $n = 3d^2 - 4 =$ 8, 23, but not 44, 71, open for $n \geq 104$.
- $t \geq 8$, then $t = 11$, $n = 24$, $|X| = 196560$, $X = \mathrm{Min}(\Lambda_{24})$ (unique)

Tight spherical designs.

Open problem.

Classify tight spherical t-designs for $t = 5$ and $t = 7$.

Conjecture.

- There are only three tight 5-designs in dimension ≥ 3:
  - The icosahedron in dimension 3,
  - $\mathrm{Min}(E_7^{\#})$ in dimension 7,
  - $\mathrm{Min}(M_{23}^{\#})$ in dimension 23.
- There are only two tight 7-designs in dimension ≥ 3:
  - $\mathrm{Min}(\mathbb{E}_8)$ in dimension 8
  - $\mathrm{Min}(O_{23})$ in dimension 23.

Tight designs and lattices

Theorem.

- Let X be a tight 5-design. Then
  - $X = -X$, $n = d^2 - 2$ with $d = 2m + 1$ odd.
  - Assume that $(x, x) = d$ for all $x \in X$. Then
  - $(x, y) \in \{\pm d, \pm 1\}$ for all $x, y \in X$.
- Let X be a tight 7-design. Then
  - $X = -X$, $n = 3d^2 - 4$. Assume that $(x, x) = d$ for all $x \in X$.
  - $(x, y) \in \{\pm d, \pm 1, 0\}$ for all $x, y \in X$.

Corollary.

$L_X := \langle X \rangle_{\mathbb{Z}}$ is an integral lattice with $\min(L_X) \leq d$.

Tight 5-designs and lattices.

$n = d^2 - 2$, $d = 2m+1$, $X \subset S^{n-1}(d)$ tight 5-design. $\Lambda := \langle X \rangle$.
Existence for $m = 1, 2$, non-existence for $m = 3, 4$.

Theorem

- Λ is an odd lattice.
- $\mathrm{Min}(\Lambda) = X$ if $m \leq 9$.
- $(x, y) \in \{\pm d, \pm 1\}$ for $x, y \in X$ (odd)
- $\Lambda_0 := \{v \in \Lambda \mid (v, v) \text{ even } \} = \langle x - y \mid x, y \in X \rangle$
- $\frac{1}{2}\Lambda_0 \subset \Lambda^{\#}$ so $\Gamma := \frac{1}{\sqrt{2}}\Lambda_0$ is integral.
- $|\Gamma^{\#}/\Gamma| = 2$ if $m + 1 \in 2\mathbb{Z} - 8\mathbb{Z}$, and $m(m+1)$ odd square free.
- $|\Gamma^{\#}/\Gamma| = 6$ if $m \in 2\mathbb{Z} - 8\mathbb{Z}$, and $m(m+1)$ odd square free.
- If $m \in 2\mathbb{Z} - 8\mathbb{Z}$, and $m(m+1)$ odd square free then $m \equiv -1 \pmod 3$.
- $m \neq 4, 6, 10, 12, 22, 28, 30, 34, 42, 46, \ldots$.

Tight 7-designs and lattices.

$n = 3d^2 - 4$, $X \subset S^{n-1}(d)$ tight 7-design. $\Lambda := \langle X \rangle$.
Existence for $d = 2, 3$, non-existence for $d = 4, 5$.

Theorem

- Λ is an integral lattice.
- Λ is even, if d is even.
- $\Lambda = \Lambda^{\#}$ if
  - $\nu_p(d^3 - d) < 3$ for all primes $p \geq 5$ and
  - $\nu_3(d^3 - d) < 4$ and
  - $\nu_2(d) < 5$.
- If $\Lambda = \Lambda^{\#}$ then $d \notin 4\mathbb{Z}$.
- $d \neq 4, 8, 12, 16, 20, 24, 28, 36, 40, 44, \ldots$.

For $d = 6$ we know

- $\Lambda \subset \mathbb{R}^{104}$ even unimodular of minimum 6.
- $X = \mathrm{Min}(\Lambda)$, $\Lambda_8 = \emptyset$.
- This determines θ_Λ.
- All layers of Λ are spherical 7-designs.

Equivalent conditions for designs

Equivalent are:

- X spherical t-design
- $\sum_{x\in X} f(x) = 0$ for all $f \in Harm_d$ and all $1 \leq d \leq t$.
- Let $\{e, o\} = \{t, t-1\}$ with e even and o odd. Then there is $c \in \mathbb{R}$ such that for all $\alpha \in \mathbb{R}^n$

$$\sum_{x\in X} (x,\alpha)^e = c(\alpha,\alpha)^{e/2}, \quad \sum_{x\in X} (x,\alpha)^o = 0.$$

$c = c(e, n, |X|) = \frac{1\cdot 3\cdot 5\cdots(e-1)|X|}{n(n+2)\cdots(n+e-2)}$

$t = 7$, $(x,x) = d$, $n = 3d^2 - 4$, $X = Y \dot\cup -Y$, $|Y| = n(n+1)(n+2)/6$, $\Lambda := \langle X\rangle$:

- $\sum_{x\in Y} (x,\alpha)^6 = \frac{5}{2} d(d^2-1)(\alpha,\alpha)^3$
- $\sum_{x\in Y} (x,\alpha)^4 = \frac{3}{2} d^2(d^2-1)(\alpha,\alpha)^2$
- $\sum_{x\in Y} (x,\alpha)^2 = \frac{1}{2} (3d^2-2)(d^2-1)d(\alpha,\alpha)$
- For $\alpha \in \Lambda^{\#}$ then rhs all integers.

Tight 7 design $X = Y \dot\cup -Y$, $\Lambda = \langle X \rangle$, $\Gamma = \Lambda^{\#}$

Theorem.

$\Lambda = \Lambda^{\#}$ if

- $\nu_p(d^3 - d) < 3$ for all primes $p \geq 5$ and
- $\nu_3(d^3 - d) < 4$ and
- $\nu_2(d) < 5$.

- Proof. Know that Λ is integral.
- So it is enough to prove that $\Lambda^{\#}$ is integral.
- $\alpha, \beta \in \Lambda^{\#} \Rightarrow (x, \beta)(x, \alpha)((x, \alpha)^2 - 1)((x, \alpha)^2 - 4) \in 120\mathbb{Z}$ so

$$\frac{d^3 - d}{240}(\alpha, \beta)(12d^2 - 8 - 15d(\alpha, \alpha) + 5(\alpha, \alpha)^2) \in \mathbb{Z}.$$

- Taking $\alpha = \beta$ we obtain $(\alpha, \alpha) \in \mathbb{Z}$.
- Then easily $(\alpha, \beta) \in \mathbb{Z}$ for arbitrary $\alpha, \beta \in \Lambda^{\#}$