Small-span characteristic polynomials of integer symmetric matrices

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PLAN

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- Future work

Integer symmetric matrices (ISMs)

These are things like:

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 3 \\ -2 & 3 & 7 \end{pmatrix}$$

(symmetric square matrix, integer entries)

Properties of ISMs

Their characteristic polynomials

- are monic

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To what extent is the converse true?

Example 1

The polynomial $x^2 - 2$ is monic, has integer coefficients, and all roots real.

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It is the characteristic polynomial of

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Example 2

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We need $a^2 + b^2 = 3$.

Example 2 (continued)

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Yes!

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

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This is not the characteristic polynomial of an ISM. (Why?)

But it is the min. poly. of the following 6×6 ISM:

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem of Estes and Guralnick (1993)

Let $f(x)$ be a monic, separable polynomial with integer coefficients, degree n, and with all roots real.

If $n \leq 4$, then f is the min. poly. of a $2n \times 2n$ ISM.

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They conjectured that the answer is 'yes'.

Answer: Dobrowolski (2008)

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- Indeed there exist infinitely many f (monic, separable, integer coefficients, and with all roots real) for which f is not the min. poly. of *any* ISM.

- He shows that if f (degree n) is the min. poly. of an ISM, then the discriminant of f is at least n^n. For large, highly composite m, the discriminant of the min. poly. of $2\cos(\pi/m)$ is too small.

Let's change the question

What is the smallest n such that there is a monic, separable polynomial $f(x)$ of degree n, with integer coefficients and with all roots real, and with f not the min. poly. of any integer symmetric matrix?

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- Dobrowolski: $5 \leq n \leq 2880$
- More precise answer: $n \in \{5, 6\}$

Some degree-6 examples

I claim that the following polyomials are monic, separable, with all roots real, but do not arise as the min. poly. of any ISMs:

- $x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1$
- $x^6 - 7x^4 + 14x^2 - 7$
- $x^6 - 6x^4 + 9x^2 - 3$

Summary to this point

We don't fully understand which polynomials arise as characteristic polynomials of integer symmetric matrices.

We don't fully understand which polynomials arise as min. polys. of integer symmetric matrices.

SMALL-SPAN POLYNOMIALS

- Definition

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- History

SMALL-SPAN: DEFINITION

A totally real, monic polynomial with integer coefficients,

$$f(x) = x^d + a_{d-1}x^{d-1} + \cdots + x_0\,,$$

with roots $\alpha_1 \le \cdots \le \alpha_d$, has span $\alpha_d - \alpha_1$.

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- For any integer c, and any $\varepsilon = \pm 1$, the polynomials $f(x)$ and $\varepsilon^d f(\varepsilon x + c)$ will be called equivalent.

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- Any small-span polynomial is equivalent to one with all its roots in the interval $[-2, 2.5)$.

SMALL-SPAN: WHY 4?

- Suppose that $f(x)$ (monic, integer coefficients, all roots real) has all its roots in the interval $[-2, 2]$. Then the roots of $f(x)$ are all of the form $2\cos(2\pi/m)$, where m is a natural number.

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- Any small-span polynomial that is not equivalent to a cosine polynomial is especially interesting.

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- Capparelli, Del Fra, Sciò (2010): up to degree 14, conjectured lists for degrees 15 and 16.
- Stop press (June 2010): Rhin et al have verified the degree 15 list.

SMALL-SPAN: SUMMARY

degree	# classes	# non-cosine
1	1	0
2	4	1
3	5	3
4	14	10
5	15	14
6	17	13
7	15	15
8	26	21

degree	# classes	# non-cosine
9	21	19
10	28	15
11	11	10
12	16	9
13	4	4
14	10	9
15	7	6
16	≥ 9	≥ 3

SMALL-SPAN CHARACTERISTIC POLYNOMIALS

We can intersect the previous two (unsolved) problems, and get an easier problem:

Which small-span polynomials arise as characteristic polynomials (or minimal polynomials) of ISMs?

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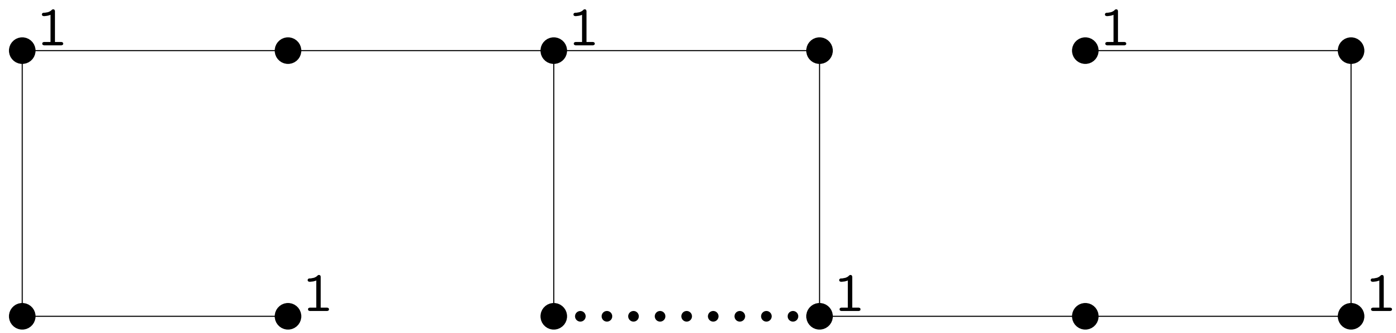
Which small-span polynomials arise as characteristic polynomials (or minimal polynomials) of ISMs?

There is a natural notion of equivalence.

SMALL-SPAN CHARACTERISTIC POLYNOMIALS: AN EXAMPLE



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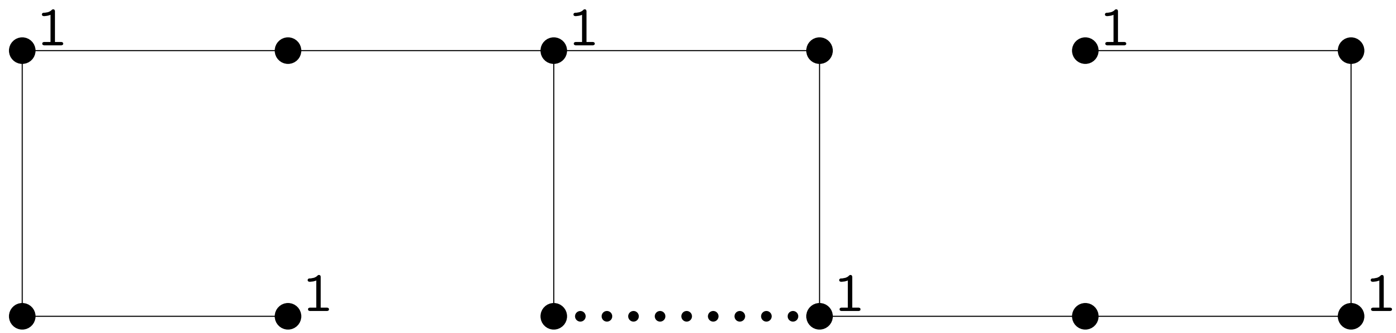

Eigenvalues:

$-1.4955\ldots,\ -1.4955\ldots,\ -1,\ -1,\ -0.2196\ldots,\ -0.2196\ldots,$
$1.2196\ldots,\ 1.2196\ldots,\ 2,\ 2,\ 2.4955\ldots,\ 2.4955\ldots\ .$

GROWING

- FACT: For $n > 1$, any small-span n-by-n ISM can be 'grown' from an $(n-1)$-by-$(n-1)$ small-span ISM.

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- GROWING ALGORITHM: find all 1-by-1 examples (up to equivalence), grow to 2-by-2, 3-by-3, etc..

RESULTS: MAXIMAL SMALL-SPAN ISMs UP TO EQUIVALENCE

n	#	#′
1	1	
2	1	
3	2	
4	21	
5	22	

n	#	#′
6	48	
7	36	
8	59	
9	25	
10	27	

n	#	#′
11	15	
12	17	
13	15	
14	16	
15	17	

RESULTS: REMOVING MEMBERS OF 10 FAMILIES

n	#	#$'$
1	1	1
2	1	1
3	2	1
4	21	19
5	22	19

n	#	#$'$
6	48	43
7	36	28
8	59	50
9	25	15
10	27	15

n	#	#$'$
11	15	2
12	17	3
13	15	0
14	16	0
15	17	0

CLASSIFICATION THEOREM

$n > 12$	$\#$	$\#'$
n	$n + 2$	0

APPLICATION: A QUESTION OF ESTES AND GURALNICK

Computations + a small argument produce lots of small-degree counterexamples to the conjecture of Estes and Guralnick concerning minimal polynomials of ISMs.

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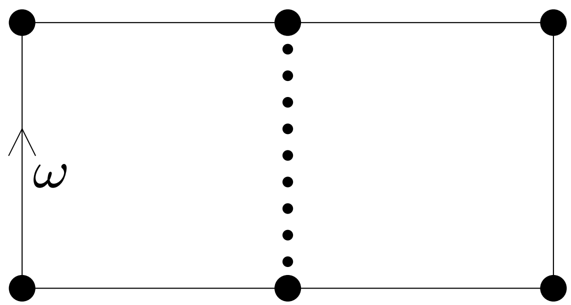
- Entries from $\mathcal{O}_K$ for various number fields K? (Hermitian)

- Gary Greaves has completed $[K : \mathbf{Q}] = 2$.

- Two of the three degree-6 polynomials now appear as minimal polynomials.

THANK YOU FOR LISTENING

$\omega = (1 + \sqrt{-3})/2$



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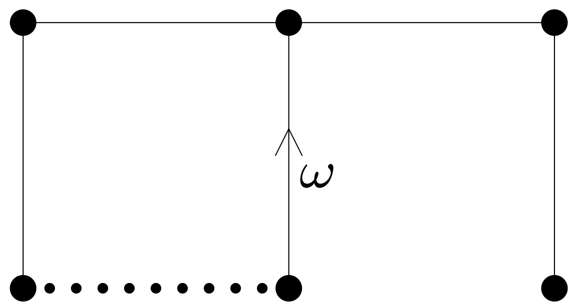

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