Weil Polynomials of K3-Surfaces

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joint work with Jörg Jahnel

K3 surfaces

Definition
A K3 surface is a simply connected proper algebraic surface with trivial canonical class.

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Examples

- A K3 surface of degree 2 is a double cover of $\mathbf{P}^2$, ramified at a smooth sextic curve.
- A K3 surface of degree 4 is a smooth quartic in $\mathbf{P}^3$.
- A K3 surface of degree 6 is a smooth complete intersection of a quadric and a cubic in $\mathbf{P}^4$.
- A K3 surface of degree 8 is a smooth complete intersection of three quadrics in $\mathbf{P}^5$.

K3 surfaces as complex algebraic surfaces

Properties of K3 surfaces

Betti numbers: 1, 0, 22, 0, 1

Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Picard group: $\mathbb{Z}^n$ for $n \in \{1, \ldots, 20\}$

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Hard problem

Compute the geometric Picard group explicitly.
(For a surface defined over $\mathbb{Q}$)
Basic strategy: Use reduction modulo p.
This leads to an injection of Picard groups $\mathrm{Pic}(V_{\overline{\mathbb{Q}}}) \hookrightarrow \mathrm{Pic}(V_{\overline{\mathbb{F}}_p})$.

K3 surfaces over finite fields

The cohomology

The second etale cohomology group has dimension 22. One could ask for the Frobenius action. This is closely related to the number of points on the surface.

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Conjecture (Tate)

The geometric Picard group generates the subspace of the second etale cohomology on which all eigenvalues are $q\zeta$. (ζ is a root of unity.)

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Remarks

- The Picard rank is predicted to be even.
- The Tate conjecture is proven for most K3 surfaces.

Artin-Tate conjecture

Notation

- V a K3 surface over $\mathbb{F}_q$
- Φ charcteristic polynomial of Frobenius on $H^2_{\text{ét}}(V_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$
- ρ rank of arithmetic Picard group
- Δ discriminant of arithmetic Picard group
- $\mathrm{Br}(V)$ the Brauer group. Order is a square (if finite).

Conjecture (Artin-Tate, special case of K3 surfaces)

$$|\Delta| = \frac{\lim_{T \to q} \frac{\Phi(T)}{(T-q)^\rho}}{q^{21-\rho} \# \mathrm{Br}(V)} .$$

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Theorem(Milne)
The Tate conjecture implies $\# \text{Br}(V) < \infty$ and the Artin-Tate conjecture.

Computing Φ

Theorem (Lefschetz Trace Formula)

$$\# V(\mathbb{F}_{q^e}) = 1 + q^{2e} + \mathrm{Tr}(\mathrm{Frob}^e)$$

Theorem (Newton's identities)
The T^{22-e}-coefficient of Φ can be computed from the traces of $\mathrm{Frob}, \mathrm{Frob}^2, \ldots, \mathrm{Frob}^e$.

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Theorem (Functional equation)

$$q^{22}\Phi(T) = \pm T^{22}\Phi\left(\frac{q^2}{T}\right)$$

Observation (Hyperplane section)

$$\Phi(q) = 0$$

Computing Φ II

Algorithm

1. Count points on $V(\mathbb{F}_q)$, $V(\mathbb{F}_{q^2})$,..., $V(\mathbb{F}_{q^{10}})$.
2. Compute the coefficient of T^{21},...,T^{12}. (Newton)
3. Apply the functional equation.
   Determine the coefficient of T^0,...,T^{10} up to a common sign.
4. Get the coefficient of T^{11} by using $\Phi(q) = 0$.

Result

Two candidates for Φ. One for each sign in the functional equation.

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ToDo

Exclude one of them.

Computing Φ III

Determination of sign (naive)
Count points on $V(\mathbb{F}_{q^{11}})$, $V(\mathbb{F}_{q^{12}})$, ... until sign is determined.

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Theorem (Deligne)
All roots are of absolute value q. (Basic strategy, used in our 2008 paper.)
I.e., we show that the roots are in general impossible as eigenvalues on the etale cohomology.

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Goal:
Derive properites of characteristic polynomials to show:

- A candidate is impossible for K3 surfaces in general.
- A candidate is impossible for K3 surfaces of the given degree.
- A candidate is impossible for K3 surfaces with known extra structure.

Second application

Bounding Picard rank (using $\#V(\mathbb{F}_q),\ldots,\#V(\mathbb{F}_{q^9})$)

1. Compute the coefficients as above for $T^{21},\ldots,T^{13},T^9,\ldots,T^0$.
2. 3 coefficients and an unknown sign remain.
3. Assume more than two zeros of the form $q\zeta$.
   I.e., we assume a Picard rank bigger than 2.
   (The order of ζ is at most 66.)
4. Compute the characteristic polynomial for each assumption.
   (Solve a linear system of equations.)
5. Exclude as many of the candidates as possible.

Result
In some cases we can prove the upper bound 2 for the geometric Picard rank, without the most costly counting step.

A comparison with other types of varieties

Theorem (Honda)
For each Weil polynomial f, there exists an abelian variety A such that $f^n = \Phi$ for some n. Here Φ is the characteristic polynomial of Frobenius on the first etale cohomology of A.

Remark
The action of Frobenius on the other cohomology groups is determined by its action on the first.

Case of curve (J.-P. Serre)
The inequalities $\#C(\mathbb{F}_q) \geq 0$ and $\#C(\mathbb{F}_{q^2}) \geq \#C(\mathbb{F}_q)$ restrict the possibilities for the characteristic polynomial. (Here q small and gen(C) big.)

Heuristics

Counting (Nicholas M. Katz)

- There are about q^{55} hypothetical characterstic polynomials.
- There are about q^{19} K3 surfaces of a given degree.
- There are not much more K3 surfaces in general.

Conclusion

The map from surfaces to polynomials can not be surjective.

Question

Can we say something about the image?

Example

Equation ($K3$ surface of degree 2 over $\mathbb{F}_7$)

$$w^2 = y^6+3z^6+5xz^5+5x^2y^4+x^2z^4+3x^3y^3+x^3z^3+5x^4y^2+x^4z^2+5x^5y+2x^6\,.$$

Point counting (up to $\mathbb{F}_{7^9}$)
66, 2 378, 118 113, 5 768 710, 282 535 041, 13 841 275 877, 678 223 852 225, 33 232 944 372 654, and 1 628 413 551 007 224

Question
Can we prove an upper bound of 2 for the Picard rank?

Hypothetical characteristic polynomials

Assuming the geometric Picard rank is bigger than 2 we get

$$\begin{aligned}\Phi_i(t) = t^{22} &- 16\,t^{21} + 140\,t^{20} - 1\,029\,t^{19} + 5\,831\,t^{18} - 36\,015\,t^{17} + 268\,912\,t^{16} \\ &- 1\,882\,384\,t^{15} + 11\,529\,602\,t^{14} - 46\,118\,408\,t^{13} + a_i t^{12} + b_i t^{11} + c_i t^{10} \\ &+ (-1)^{j_i}[-110\,730\,297\,608\,t^9 + 1\,356\,446\,145\,698\,t^8 - 10\,851\,569\,165\,584\,t^7 \\ &\quad + 75\,960\,984\,159\,088\,t^6 - 498\,493\,958\,544\,015\,t^5 + 3\,954\,718\,737\,782\,519\,t^4 \\ &\quad - 34\,196\,685\,556\,119\,429\,t^3 + 227\,977\,903\,707\,462\,860\,t^2 \\ &\quad - 1\,276\,676\,260\,761\,792\,016\,t + 3\,909\,821\,048\,582\,988\,049]\end{aligned}$$

for

$$\begin{aligned} j_1 &= 0, & (a_1, b_1, c_1) &= (161\,414\,428, -1\,129\,900\,996, \quad 7\,909\,306\,972)\,, \\ j_2 &= 1, & (a_2, b_2, c_2) &= (\ 80\,707\,214, \qquad 0, -3\,954\,653\,486)\,, \\ j_3 &= 1, & (a_3, b_3, c_3) &= (121\,060\,821, \qquad 0, -5\,931\,980\,229)\,. \end{aligned}$$

All roots are of absolute value 7.

Application of the Artin-Tate formula

Result

polynomial	field	arithmetic Picard rank	$\#\mathrm{Br}(V)\lvert\Delta\rvert$
Φ_1	$\mathbb{F}_7$	2	58
	$\mathbb{F}_{49}$	2	4524
Φ_2	$\mathbb{F}_7$	1	4
	$\mathbb{F}_{49}$	2	1996
Φ_3	$\mathbb{F}_7$	1	6
	$\mathbb{F}_{49}$	2	2997

Interpretation

Φ_1 is impossible in general, Φ_2, Φ_3 are impossible in degree 2.

Conclusion

The geometric Picard rank is at most 2.

Computation

$\# V(\mathbb{F}_{7^{10}}) = 79\,792\,267\,067\,823\,523$

Resulting polynomials

$$\begin{aligned}\Phi_i(t) = {} & t^{22} - 16\,t^{21} + 140\,t^{20} - 1\,029\,t^{19} + 5\,831\,t^{18} - 36\,015\,t^{17} + 268\,912\,t^{16} \\ & - 1\,882\,384\,t^{15} + 11\,529\,602\,t^{14} - 46\,118\,408\,t^{13} + 40\,353\,607\,t^{12} + a_i t^{11} \\ & + (-1)^{j_i} [\, -1\,977\,326\,743\,t^{10} + 110\,730\,297\,608\,t^{9} - 1\,356\,446\,145\,698\,t^{8} \\ & + 10\,851\,569\,165\,584\,t^{7} - 75\,960\,984\,159\,088\,t^{6} + 498\,493\,958\,544\,015\,t^{5} \\ & - 3\,954\,718\,737\,782\,51\,9 t^{4} + 34\,196\,685\,556\,119\,429\,t^{3} \\ & - 227\,977\,903\,707\,462\,860\,t^{2} + 1\,276\,676\,260\,761\,792\,016\,t \\ & - 3\,909\,821\,048\,582\,988\,049]\end{aligned}$$

for $j_4 = 0$, and $a_4 = 0$, or $j_5 = 1$, and $a_5 = 564\,950\,498$.
All roots are of absolute value 7.

Application of the Artin-Tate formula

Result

polynomial	field	arithmetic Picard rank	$\#\mathrm{Br}(V)\lvert\Delta\rvert$
Φ_4	$\mathbb{F}_7$	1	2
	$\mathbb{F}_{49}$	2	997
Φ_5	$\mathbb{F}_7$	2	55
	$\mathbb{F}_{49}$	2	4125

Interpretation

Φ_4 is possible for a K3 surface of degree 2.

Φ_5 is impossible for K3 surfaces in general.

Conclusion

Φ_4 is the characteristic polynomial.

The minus-sign in the functional equation is correct.

The methods used

Using known divisors

If we know parts of the Picard group then we can compare known and predicted ranks. In the case of equal ranks, we can compare the discriminants.

Without a known divisor

We can compare the predicted Picard rank for $V(\mathbb{F}_q)$ and $V(\mathbb{F}_{q^d})$. In the case of equal ranks, we can compare the discriminants.

We call this the **field extension condition**.

Remark

This means: The Artin-Tate formula contradicts itself under field extension.

The field extension condition in detail

Theorem (Jahnel & E. 2010)
The field extension condition is independent of the Tate conjecture.

We can use arguments from Milne's proof.

Theorem (Jahnel & E. 2010)
The field extension condition for $\mathbb{F}_{q^2}/\mathbb{F}_q$ implies all other field extension conditions.

Theorem (Jahnel & E. 2010)
Let $\Phi(t) = (t-q)^r \psi(t)$ with $\psi(q) \neq 0$.
The field extension condition means $q\psi(-q)$ is a square in $\mathbb{Q}$.

A statistical test of the conditions

The sample

	$p = 2$	$p = 3$	$p = 5$	$p = 7$
$d = 2$	1000 rand	1000 rand	1000 dec	1000 dec
$d = 4$	1000 rand	1000 ell		
$d = 6$	1000 rand	1000 ell		
$d = 8$	1000 rand	1000 ell		

dec = decoupled, ell = elliptic, rand = random

Point counting

- naive counting
- using elliptic fibration (if exists)
- decoupled case (convolution)

Proving geometric Picard rank ≤ 2 using data up to $\mathbb{F}_{q^9}$

	Number of polynomials	0	1	2	3	4	5	6
$d = 2,\ p = 2$	without	84	479	312	89	21	12	3
	with A-T conditions	149	598	218	28	7	0	0
$d = 2,\ p = 3$	without	116	480	285	88	24	4	3
	with A-T conditions	214	573	193	20	0	0	0
$d = 2,\ p = 5$	without	85	581	209	96	25	4	0
	with A-T conditions	158	651	169	20	2	0	0
$d = 2,\ p = 7$	without	92	534	232	98	37	7	0
	with A-T conditions	214	611	154	21	0	0	0
$d = 4,\ p = 2$	without	40	532	303	87	29	8	1
	with A-T conditions	81	638	249	27	5	0	0
$d = 4,\ p = 3$	without	22	669	242	57	9	1	0
	with A-T conditions	53	785	161	1	0	0	0
$d = 6,\ p = 2$	without	39	549	312	70	22	6	2
	with A-T conditions	83	645	257	14	1	0	0
$d = 6,\ p = 3$	without	16	713	217	47	7	0	0
	with A-T conditions	50	797	148	5	0	0	0
$d = 8,\ p = 2$	without	25	657	268	38	8	4	0
	with A-T conditions	29	723	239	5	4	0	0
$d = 8,\ p = 3$	without	12	720	236	27	4	1	0
	with A-T conditions	20	803	175	2	0	0	0

Decision of sign using data up to $\mathbb{F}_{q^{10}}$

p	2	3	5	7	2	3	2	3	2	3
d	2	2	2	2	4	4	6	6	8	8
Known signs without A-T	768	843	864	869	761	876	790	888	822	897
Known signs using A-T	863	940	940	961	863	943	868	933	867	944
Remaining unknown signs	137	60	60	39	137	57	132	67	133	56
Data up to $\mathbb{F}_{p^{11}}$ insufficient	84	23	15	12	69	19	77	25	72	21
Data up to $\mathbb{F}_{p^{12}}$ insufficient	41	11	2	1	39	3	42	11	47	7
Data up to $\mathbb{F}_{p^{13}}$ insufficient	22	5	1	0	24	2	20	2	24	2
Data up to $\mathbb{F}_{p^{14}}$ insufficient	13	2	0	0	12	0	13	1	8	0
Data up to $\mathbb{F}_{p^{15}}$ insufficient	7	0	0	0	8	0	7	0	5	0
Data up to $\mathbb{F}_{p^{16}}$ insufficient	4	0	0	0	3	0	2	0	4	0
Data up to $\mathbb{F}_{p^{17}}$ insufficient	4	0	0	0	2	0	2	0	0	0
Data up to $\mathbb{F}_{p^{18}}$ insufficient	4	0	0	0	0	0	1	0	0	0
Data up to $\mathbb{F}_{p^{19}}$ insufficient	2	0	0	0	0	0	1	0	0	0
Data up to $\mathbb{F}_{p^{20}}$ insufficient	0	0	0	0	0	0	0	0	0	0

Proving geometric Picard rank ≤ 2 using data up to $\mathbb{F}_{q^{10}}$

		rank 2 proven not using $\#V(\mathbb{F}_{p^{10}})$	rank 2 proven	rank 2 possible
$p = 2,\ d = 2$	without	84	271	330
	with A-T conditions	149	278	301
$p = 3,\ d = 2$	without	116	397	460
	with A-T conditions	214	409	428
$p = 5,\ d = 2$	without	85	353	425
	with A-T conditions	158	360	382
$p = 7,\ d = 2$	without	92	460	511
	with A-T conditions	214	464	476
$p = 2,\ d = 4$	without	40	132	197
	with A-T conditions	81	138	163
$p = 3,\ d = 4$	without	22	79	114
	with A-T conditions	53	79	81
$p = 2,\ d = 6$	without	39	145	183
	with A-T conditions	83	152	163
$p = 3,\ d = 6$	without	16	74	101
	with A-T conditions	50	74	81
$p = 2,\ d = 8$	without	25	65	93
	with A-T conditions	29	65	74
$p = 3,\ d = 8$	without	12	23	47
	with A-T conditions	20	23	25

Summary

Idea
The Artin-Tate formula implies restrictions for the characteristic polynomial of the Frobenius on the second etale cohomology.

Practical
The derived conditions can be checked easily.

Result
Our method is independent of the Tate conjecture.
We halve the cases of unknown sign.
We double the cases with rank ≤ 2 proven using only data up to $\mathbb{F}_{q^9}$.