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The Hodge and Tate conjectures: some numerical experiments.

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July 23, 2010

Algebraic cycles and the cycle class map

V= smooth, projective variety over $\mathbb{C}$ of dimension d.

$$\mathrm{CH}^j(V) = \left\{ \begin{array}{c} \text{Codimension } j \\ \\ \text{algebraic cycles on } V \end{array} \right\} \otimes \mathbb{Q} \ / \ \sim,$$

where $\sim$ denotes rational equivalence.

The cycle class map:

$$\mathrm{cl} : \mathrm{CH}^j(V) \longrightarrow H^{j,j}_{\mathrm{dR}}(V(\mathbb{C})) \cap H^{2j}_B(V(\mathbb{C}), \mathbb{Q}).$$

$$\langle \mathrm{cl}(\Delta), \alpha \rangle = \int_{\Delta(\mathbb{C})} \alpha, \qquad \forall \alpha \in H^{2d-2j}_{\mathrm{dR}}(V(\mathbb{C})).$$

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The Hodge conjecture

Conjecture (Hodge Conjecture)

The cycle class map cl *is surjective.*

A cohomology class in $H^{j,j}_{\mathrm{dR}}(V(\mathbb{C})) \cap H^{2j}_{B}(V(\mathbb{C}), \mathbb{Q})$

—i.e, a class of type (j, j) with rational periods—

is called a *Hodge cycle*.

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Every Hodge cycle is the class of an algebraic cycle.

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The Tate conjecture

The cycle class map has an analogue in ℓ-adic étale cohomology:

$$\mathrm{cl}_\ell : \mathrm{CH}^j(V/F) \otimes \mathbb{Q}_\ell \longrightarrow H^{2j}_{\mathrm{et}}(V_{\bar{F}}, \mathbb{Q}_\ell)(j)^{G_F}.$$

Conjecture (Tate)

The ℓ-adic cycle class map is surjective.

The Hodge conjecture is known for surfaces, and for codimension one cycles, but there seems to be very little evidence for cycles of higher codimension.

The challenge

André Weil (Collected works, 1979).

> *La question que pose la "conjecture de Hodge" est bien naturelle... Par malheur, en dépit du mot de "conjecture", il n'y a, que je sache, pas l'ombre d'une raison d'y croire; on rendrait service aux géomètres si l'on pouvait trancher la question au moyen d'un contre-exemple.*

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CM elliptic curves

$$D \in \{7, 11, 19, 43, 67, 163\}.$$

K = quadratic imaginary field of discriminant $-D$.

(K has class number one and $\mathcal{O}_K^\times = \pm 1$.)

A = elliptic curve over $\mathbb{Q}$ of conductor D^2 with $\mathrm{End}_K(A) = \mathcal{O}_K$.

$\omega_A \in \Omega^1(A/\mathbb{Q})$: the Néron diferential of A.

$$\Lambda_A := \{\int_\gamma \omega_A\} = \mathcal{O}_K \cdot \Omega_A.$$

$\Omega_A \in \mathbb{C}$ is called the *Chowla-Selberg period* attached to A.

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The elliptic curve A and its Chowla-Selberg period

D	a_1	a_2	a_3	a_4	a_6	Ω_A
7	1	-1	0	-107	552	$1.93331170\ldots$
11	0	-1	1	-7	10	$4.80242132\ldots$
19	0	0	1	-38	90	$4.19055001\ldots$
43	0	0	1	-860	9707	$2.89054107\ldots$
67	0	0	1	-7370	243528	$2.10882279\ldots$
163	0	0	1	-2174420	1234136692	$0.79364722\ldots$

The de Rham cohomology of A

Because A has CM, we can write

$$H^1_{\mathrm{dR}}(A/K) = K \cdot \omega_A \oplus K \cdot \eta_A,$$

where

1. ω_A is the Néron differential, viewed as an element of $\Omega^1(A/K)$;
2. η_A is the generator of $H^{0,1}_{\mathrm{dR}}(A/\mathbb{C})$ normalised so that

$$\langle \omega_A, \eta_A \rangle \left(= \frac{1}{2\pi i} \int_{A(\mathbb{C})} \omega_A \wedge \eta_A \right) = 1.$$

The fact that η_A is defined over K is specific to the CM setting.

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The Hecke character attached to A

Theorem (Deuring)

There is a Hecke character ψ_A of K of infinity type $(1, 0)$ and conductor $\sqrt{-D}$ satisfying $L(A, s) = L(\psi_A, s)$.

More precisely, for all $a \in \mathcal{O}_K$,

$$\psi_A((a)) = \left(\frac{a}{\sqrt{-D}}\right) a.$$

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Hecke characters of higher weight and theta series

Fix an integer $r \geq 0$, and let $\psi = \psi_A^{r+1}$.

The character ψ is of infinity type $(r+1, 0)$.

$$\text{conductor}(\psi) = \begin{cases} (\sqrt{-D}) & \text{if } r \text{ is even;} \\ 1 & \text{if } r \text{ is odd.} \end{cases}$$

$$\theta_\psi = \frac{1}{2} \sum_{a \in \mathcal{O}_K} \psi(a) q^{a\bar{a}} \in \begin{cases} S_{r+2}(\Gamma_0(D^2)) & \text{if } r \text{ is even;} \\ S_{r+2}(\Gamma_0(D), \varepsilon_D) & \text{if } r \text{ is odd.} \end{cases}$$

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Kuga-Sato varieties

Modular forms in $S_{r+2}(\Gamma)$ give rise to cohomology classes on certain *Kuga-Sato varieties*.

Modular curve:

$$C = \begin{cases} Y_0(D^2) & \text{if } r \text{ is even;} \\ Y_0(D) & \text{if } r \text{ is odd.} \end{cases}$$

$W_r' = r$-fold fiber product of the "universal" elliptic curve over C.

The open variety W_r' admits a smooth compactification, called the *Kuga-Sato variety* attached to (C, r), and denoted W_r.

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Complex description

Let

$$\Gamma = \begin{cases} \Gamma_0(D^2) & \text{if } r \text{ is even;} \\ \Gamma_0(D) & \text{if } r \text{ is odd,} \end{cases}$$

so that $C(\mathbb{C}) = \mathcal{H}/\Gamma$. Then

$$W_r'(\mathbb{C}) = (\mathbb{Z}^{2r} \rtimes \Gamma)\backslash(\mathbb{C}^r \times \mathcal{H}),$$

where

$$(m_1, n_1, \ldots, m_r, n_r)(w_1, \ldots, w_r, \tau) = (w_1 + m_1 + n_1\tau, \ldots, w_r + m_r + n_r\tau, \tau),$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (w_1, \ldots, w_r, \tau) = \left(\frac{w_1}{c\tau + d}, \ldots, \frac{w_r}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right).$$

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The de Rham cohomology of Kuga-Sato varieties

The holomorphic $(r+1)$-form on $\mathbb{C}^r \times \mathcal{H}$

$$\omega_{\theta_\psi} = (2\pi i)^{r+1} \theta_\psi(\tau) dw_1 \cdots dw_r d\tau$$

is invariant under $\mathbb{Z}^{2r} \rtimes \Gamma$, and hence corresponds to a regular $(r+1)$-form in $\Omega^{r+1}(W_r'/\mathbb{C})$.

It extends to W_r (cuspidality) and is defined over $\mathbb{Q}$ (q-expansion principle.)

Let $[\omega_{\omega_\psi}]$ denote its class in $H^{r+1,0}_{\mathrm{dR}}(W_r/\mathbb{Q})$.

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A family of Hodge cycles

The ambient variety: $V_r := W_r \times A^{r+1}$, (of dimension $2r+2$).

Proposition

The class $\Phi_{\text{Hodge}} := \omega_{\theta_\psi} \wedge \eta_A^{r+1}$ is a Hodge cycle in $H^{2r+2}_{\text{dR}}(V_r/\mathbb{C})$.

This result is "consistent" with the ℓ-adic picture:

- The ℓ-adic representation V_{θ_ψ} of $G_{\mathbb{Q}}$ attached to the modular form θ_ψ is a constitutent of $H^{r+1}_{\text{et}}(W_{r/\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$.
- The representation V_{θ_ψ} is also isomorphic to

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The class $\Phi_{\mathrm{Hodge}} := \omega_{\theta_\psi} \wedge \eta_A^{r+1}$ is a Hodge cycle in $H^{2r+2}_{\mathrm{dR}}(V_r/\mathbb{C})$.

This result is "consistent" with the ℓ-adic picture:

- The ℓ-adic representation V_{θ_ψ} of $G_{\mathbb{Q}}$ attached to the modular form θ_ψ is a constitutent of $H^{r+1}_{\mathrm{et}}(W_{r/\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$.
- The representation V_{θ_ψ} is also isomorphic to

$$\mathrm{Ind}_K^{\mathbb{Q}} \psi_A^{r+1} \subset H^{r+1}_{\mathrm{et}}(A^{r+1}_{/\bar{\mathbb{Q}}}, \mathbb{Q}_\ell).$$

- These isomorphisms yield a non-trivial Tate cycle in $H^{2r+2}_{\mathrm{et}}(V_{r/\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)(r+1)^{G_{\mathbb{Q}}}$.

A special case of the Hodge conjecture

Question

Given D and $r \geq 0$, produce an algebraic cycle

$$\Phi \subset V_r = W_r \times A^{r+1} \quad \textit{such that } \mathrm{cl}(\Phi) = \Phi_{\mathrm{Hodge}}$$

...or show that no such cycle exists!

The Tate conjecture suggests that the cycle class Φ can be defined over K. (Or even, with some care, over $\mathbb{Q}$...)

Some examples

- $r = 0$. Then, $V_r = C \times A$. The cycle Φ is the graph of a modular parametrisation $X_0(D^2) \longrightarrow A$.
- $r = 1, D = 7$. The Kuga-Sato variety W_1 is an elliptic K3-surface with maximal Picard rank 20.
  Shioda-Inose: there exists an involution ι on W_1, such that

  $$W_1/\iota = \text{Kummer}(B) = B/\pm 1, \quad \text{with } B \sim A \times A.$$

  The desired cycle can be built from the image of

  $$W_1 \times_{W_1/\iota} B \quad \text{in} \quad W_1 \times A^2.$$

  An explicit equation (over $\mathbb{Q}$) for the Shioda-Inose structure in this case has been computed by Elkies...
- These are the only cases for which the algebraic cycle has been produced! $D = 163, r = 15$???

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Exotic modular parametrisations

Let $X_r = W_r \times A^r$, and let

$$\mathrm{CH}^{r+1}(X_r)_0 \subset \mathrm{CH}^{r+1}(X_r)$$

denote the subgroup of classes of null-homologous cycles.

Key Remark: The conjectural algebraic cycle

$$\Phi \subset V_r = W_r \times A^{r+1} = X_r \times A$$

gives rise to an *exotic modular parametrisation*

$$\Phi : \mathrm{CH}^{r+1}(X_r)_0 \longrightarrow \mathrm{CH}^1(A)_0 = A,$$

which converts null-homologous algebraic cycles into rational points on A and respects fields of definition.

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Definition of Φ

1. Let $\pi_A : X_r \times A \longrightarrow A, \quad \pi_X : X_r \times A \longrightarrow X_r$ be the natural projections to each factor.
2. If $\Delta \in \mathrm{CH}^{r+1}(X_r)$, then

$$\dim(\pi_X^{-1}(\Delta)) = \dim(\Phi) = r + 1 = \frac{1}{2} \dim V_r;$$

3. Therefore $\pi_X^{-1}(\Delta) \cdot \Phi$ (if the intersection is transversal...) is 0-dimensional, and

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Test cycles

Proposition

The group $\mathrm{CH}^{r+1}(X_r)_0(\bar{\mathbb{Q}})$ *is infinitely generated.*

Proof.

$\mathrm{CH}^{r+1}(X_r)_0$ contains an infinite collection of explicit null-homologous cycles: the *generalised Heegner cycles*. *C. Schoen*: they generate a subgroup of infinite rank. □

Generalised Heegner cycles: Indexed by $\varphi : A \longrightarrow A'$.

$$\Delta'_{\varphi,r} := \mathrm{graph}(\varphi)^r \subset (A \times A')^r = (A')^r \times A^r \subset W_r \times A^r = X_r.$$

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The conjecture on generalised Heegner cycles

The cycle $\Delta_{\varphi,r}$ is defined over a ring class field H_φ attached to φ.

Proposition

If the cycle Φ exists, then $\{\Phi(\Delta_{\varphi,r})\}_\varphi$ generates an infinite rank subgroup of $A(K^{\mathrm{ab}})$.
In particular, $\Phi(\Delta_{1,r})$ belongs to $A(K)$.

The points $\Phi(\Delta_{\varphi,r})$ are called *Chow-Heegner points*.

Some problems:

1. Calculate Chow Heegner points numerically *without* calculating the algebraic cycle Φ beforehand.

2. Describe the exact position of $\Phi(\Delta_{1,r})$ in $A(K)$ as $r \geq 0$ varies.

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Complex Abel-Jacobi maps

Recall the classical Abel-Jacobi map:

$$\mathrm{AJ}_A : \mathrm{CH}^1(A(\mathbb{C}))_0 = A(\mathbb{C}) \longrightarrow \frac{\Omega^1(A/\mathbb{C})^\vee}{H_1(A(\mathbb{C}),\mathbb{Z})} = \mathbb{C}/\Lambda_A,$$

given by

$$\mathrm{AJ}_A(\Delta)(\omega) := \int_{\partial^{-1}(\Delta)} \omega.$$

(If $\Delta = P - Q$, then $\partial^{-1}(\Delta)$ is any path from Q to P.)

Higher-dimensional analogue (Griffiths-Weil):

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If Φ exists, then

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Proof.

Functoriality of Abel-Jacobi maps under correspondences:

$$\mathrm{AJ}_A(\Phi(\Delta_{\varphi,r}))(\omega_A) = \mathrm{AJ}_{X_r}(\Delta_{\varphi,r})(\Phi^*\omega_A),$$

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An explicit complex formula

Proposition

Let $\Delta_{\varphi,r}$ be the generalised Heegner cycle on X_r corresponding to the isogeny $\varphi : A \longrightarrow A'$, and let $\tau \in \mathcal{H}$ satisfy $A'(\mathbb{C}) = \mathbb{C}/\langle 1, \tau\rangle$. Then

$$\mathrm{AJ}_{X_r}(\Delta_{\varphi,r})(\omega_{\theta_\psi} \wedge \eta_A'^r) = \Omega_A^{-r} \frac{(2\pi i)^{r+1}}{(\tau - \bar{\tau})^r} \int_{i\infty}^{\tau} (z - \bar{\tau})^r \theta_\psi(z) dz.$$

Since θ_ψ is a modular form of weight $r+2$, the expression on the right is an "incomplete Eichler integral".

One recovers the familiar complex-analytic formula for calculating Heegner points, when $r = 0$.

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The position of $\Phi(\Delta_{\varphi,r})$ in $A(K)$

Conjecture (Bertolini-Prasanna-D)

If $D = 7$, then $\Phi(\Delta_{1,r}) = 0$. For all $D \in \{11, 19, 43, 67, 163\}$ and all odd $r \geq 1$, the Chow-Heegner point $\Phi(\Delta_{1,r})$ belongs to $A(K) \otimes \mathbb{Q}$ and is given by the formula

$$\Phi(\Delta_{1,r}) = \sqrt{-D} \cdot m_r \cdot P_A,$$

where $m_r \in \mathbb{Z}$ satisfies the formula

$$m_r^2 = \frac{2r!(2\pi\sqrt{D})^r}{\Omega_A^{2r+1}} L(\psi_A^{2r+1}, r+1),$$

and P_A is the generator of $A(\mathbb{Q}) \otimes \mathbb{Q}$ given in the next slide.

The Mordell-Weil generators P_A

D	a_1	a_2	a_3	a_4	a_6	P_A
7	1	-1	0	-107	552	$-$
11	0	-1	1	-7	10	$(4, 5)$
19	0	0	1	-38	90	$(0, 9)$
43	0	0	1	-860	9707	$(17, 0)$
67	0	0	1	-7370	243528	$(\frac{201}{4}, \frac{-71}{8})$
163	0	0	1	-2174420	1234136692	$(850, -69)$

Where does this conjecture come from?

p-adic methods

- The complex Abel-Jacobi map admits a p-adic analogue:

$$\mathrm{AJ}_{X_r}^{(p)} : \mathrm{CH}^{r+1}(X_r)_0(\mathbb{C}_p) \longrightarrow \mathrm{Fil}^{r+1} H_{dR}^{2r+1}(X_r/\mathbb{C}_p)^\vee.$$

- **Bertolini, Prasanna, D**: A p-adic Gross-Zagier formula relating $\mathrm{AJ}_{X_r}^{(p)}(\Delta_{\varphi,r})$ to p-adic L-functions.
- In the case at hand, this formula gives (for all odd $r \geq 1$)

$$\frac{\mathrm{AJ}_{X_r}^{(p)}(\Delta_{1,r})(\omega_{\theta_\psi} \wedge \eta_A^r)}{\mathrm{AJ}_{X_1}^{(p)}(\Delta_{1,r})(\omega_{\theta_\psi} \wedge \eta_A^r)} = \frac{m_r}{m_1},$$

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The outcome of the experiment

Conjecture BDP was tested numerically to 200 digits of decimal accuracy, for all D and all odd $1 \leq r \leq 15$.

Let $\tilde{P}_A \in \mathbb{C}$ be a lift of $P_A \in A(\mathbb{C}) = \mathbb{C}/\Lambda_A$.

The table in the next slide reproduces an integer m_r (of relatively small height) satisfying

$$\mathrm{AJ}_{X_r}(\Delta_{1,r}) = \sqrt{-D} \cdot m_r \cdot \tilde{P}_A \pmod{\Lambda_A},$$

to within the calculated accuracy.

The numbers are in perfect agreement with a table of "square roots" of $L(\psi_A^{2r+1}, r+1)$ produced by Villegas...

The outcome of the experiment

Conjecture BDP was tested numerically to 200 digits of decimal accuracy, for all D and all odd $1 \leq r \leq 15$.

Let $\tilde{P}_A \in \mathbb{C}$ be a lift of $P_A \in A(\mathbb{C}) = \mathbb{C}/\Lambda_A$.

The table in the next slide reproduces an integer m_r (of relatively small height) satisfying

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	11	19	43	67	163
1	1	1	1	1	1
3	2	6	36	114	2172
5	−8	−16	440	6920	3513800
7	14	−186	−19026	−156282	3347376774
9	304	4176	−8352	−34999056	−238857662304
11	−352	−33984	33708960	3991188960	−3941159174330400
13	76648	545064	−2074549656	46813903656	1904546981028802344
15	274736	40959504	47714214240	−90863536574160	828743785015597346448

Conclusion

All the numerical experiments are consistent with the existence of an algebraic cycle $\Phi \in \mathrm{CH}^{r+1}(V_r)$ attached to Φ_{Hodge}.

This gives some indirect evidence for the Hodge conjecture for some varieties of large dimension (up to 32, in the computed ranges.)

Should one believe more firmly in the Hodge conjecture because of this?

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Thank you for your attention!