Improvements to ideal class group and regulator computation in real quadratic number fields

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ANTS IX

Motivations

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We achieve the computation of $\mathrm{Cl}(\Delta)$ and R_Δ for a 110-digit discriminant.

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3. Practical improvements

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- If $\Delta < 0$: **imaginary** case. If $\Delta > 0$: **real case**.
- The *fractional ideals* $\mathfrak{a}$ are the sets of the form

$$\frac{1}{d}\mathfrak{a}', \mid \; d \in \mathbb{K}, \;\; \mathfrak{a}' \text{ is an ideal of } \mathcal{O}_\Delta.$$

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Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}(\Delta)$, then we denote by $\mathfrak{a} \sim \mathfrak{b}$:

$$[\mathfrak{a}] = [\mathfrak{b}] \in \mathrm{Cl}(\Delta) \Leftrightarrow \exists \alpha \in \mathbb{K},\ \mathfrak{a} = (\alpha)\mathfrak{b}.$$

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Every unit ε satisfies $\exists n, \log|\varepsilon| = nR_\Delta$.

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3. Perform a *linear algebra phase* on M.

Complexity

We define the *subexponential* function by

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Our problems for qudratic number fields have complexity

$$L_\Delta(1/2, c),$$

where c depends on the linear algebra phase.

The factor base

We fill the factor base with invertible **prime ideals** $\mathfrak{p}$. There is p prime such that

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Under ERH, if $B > 6 \log^2 |\Delta|$, then $\mathcal{B}$ generates $\mathrm{Cl}(\Delta)$, and the lattice $\mathcal{L}$ of the relations satisfies

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We have ($\mathfrak{a}$ is $\mathcal{B}$-smooth)$\Leftrightarrow$($\mathcal{N}(\mathfrak{a})$ is B-smooth).

Hermite Normal Form

Invertible operations on rows lead to the **Hermite Normal Form** H of M :

$$H = \left(\begin{array}{ccc|cc} h_{1,1} & \dots & 0 & & \\ \vdots & \ddots & \vdots & \multicolumn{2}{c}{(0)} \\ * & \dots & h_{l,l} & & \\ \hline & & & 1 & (0) \\ & (*) & & & \ddots \\ & & & (0) & 1 \\ \hline \multicolumn{5}{c}{(0)} \end{array}\right),$$

where $\forall i > j : 0 \leq h_{ij} < h_{jj}$.

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Upper left : **Essential part**

Smith Normal Form (SNF) and class group structure

Any matrix $A \in \mathbb{Z}^{n\times n}$ with non zero determinant can be written as :

$$A = V^{-1} \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & d_n \end{pmatrix} U^{-1}$$

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If (d_i) are the diagonal coefficients of the SNF of the essential part of H then

$$Cl(\Delta) = \bigoplus_{1\leq i\leq n} (\mathbb{Z}/d_i\mathbb{Z})$$

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In fact MAGMA is much faster :($\Rightarrow$ room for improvement.

Regulator computation

Let $M = (m_{ij}) \in \mathbb{Z}^{N' \times N}$, be the relation matrix, $\mathcal{B} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_N\}$, and

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Each kernel vector of M yields a multiple of R_Δ. We recover R_Δ by successive real-GCD computation.

Relation collection via sieving

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(JacWil09) There is an ideal $\mathfrak{b}$ such that $(\gamma) = \mathfrak{a}'\mathfrak{b}$ (that is $\mathfrak{a} \cdot \mathfrak{b} \sim 1$) and

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The quadratic sieve

Let $\phi_{\mathfrak{a}}(X, Y) = aX^2 + bXY + cY^2$ and B defining $\mathcal{B}$. We look for B-smooth values of $\phi_{\mathfrak{a}}(X, Y)$. (Jac99) : use the **quadratic sieve**

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3. For $x = r_p + kp \in [-M, M]$ do $S[x] \leftarrow S[x] + \log p$ because

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4. For "large" $S[x]$, test the smoothness of $\phi_{\mathfrak{a}}(x,1)$.

Large prime variants

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- **Single large prime variant**. We authorize relations of the form

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- **Double large prime variant** . We authorise relations of the form
$$\mathfrak{a} = \underbrace{\mathfrak{p}_1 \dots \mathfrak{p}_n}_{\in \mathcal{B}} \mathfrak{p}\mathfrak{p}',$$
where $B \leq \mathcal{N}(\mathfrak{p}), \mathcal{N}(\mathfrak{p}') \leq B_2$.

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- Takes non negative $x_1, \dots, x_K$ and primes $p_1, \dots, p_N$.

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- uses a tree structure.

Relation collection timings

TAB.: Comparison of the relation collection time for $\Delta = -4(10^n + 1)$

n	0LP	1LP	2LP	2LP Batch
40	0.69	0.56	0.59	0.66
45	7.25	3.77	3.83	4.41
50	18.82	9.30	9.84	6.82
55	152.28	74.78	55.99	36.49
60	333.26	166.88	140.79	83.06
65	2033.97	853.27	478.57	368.31
70	2828.92	1277.94	822.39	670.63
75	14811.70	6033.89	3324.61	2732.68

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We describe a method for managing the growth of the density and the size of the coefficients during the elimination.

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Row $R \longrightarrow$ cost function $COST(R)$ taking into account :

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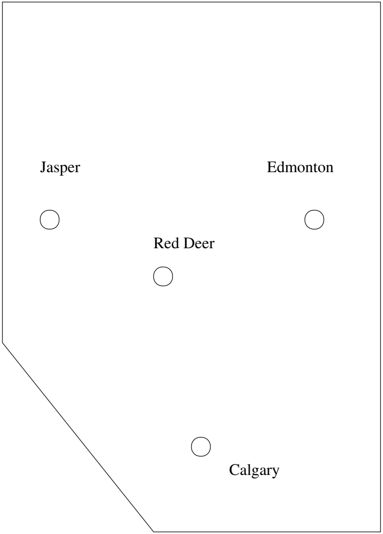
We then construct the minimum spanning tree of $\mathcal{G}$ and eliminate rows from the leaves to the root.

Minimum spanning tree on Alberta's map



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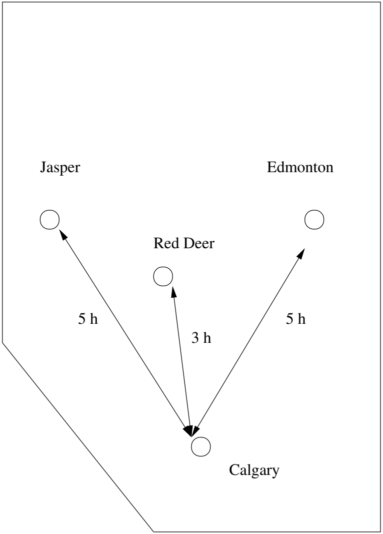

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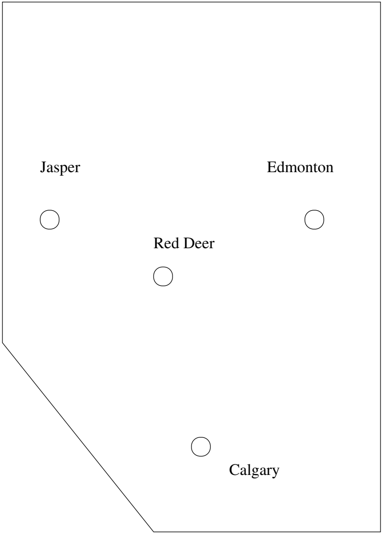

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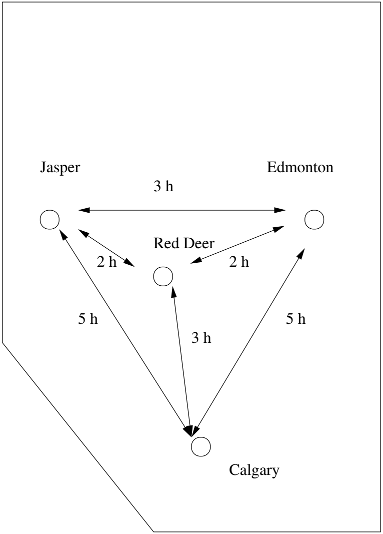

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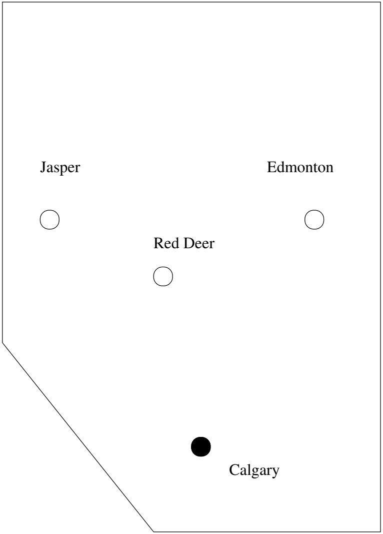

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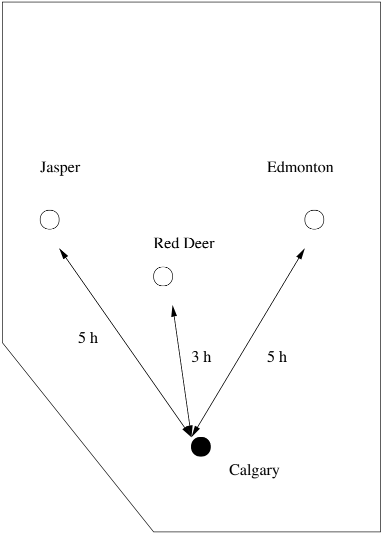

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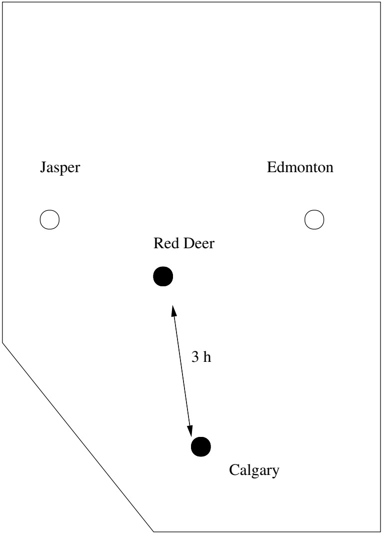

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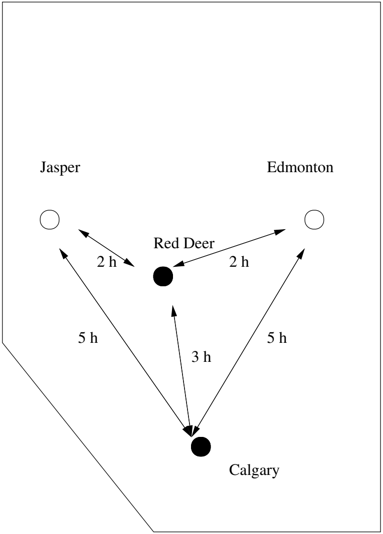

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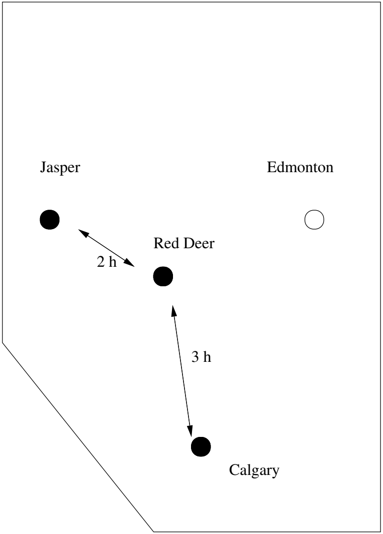

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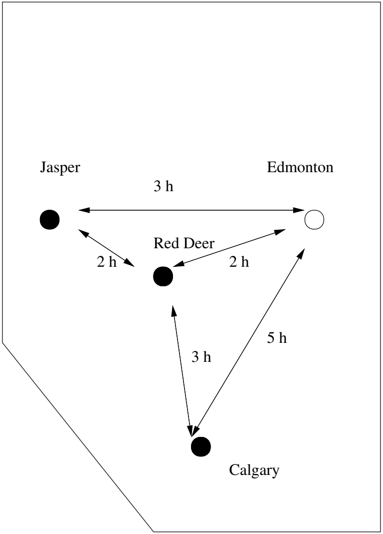

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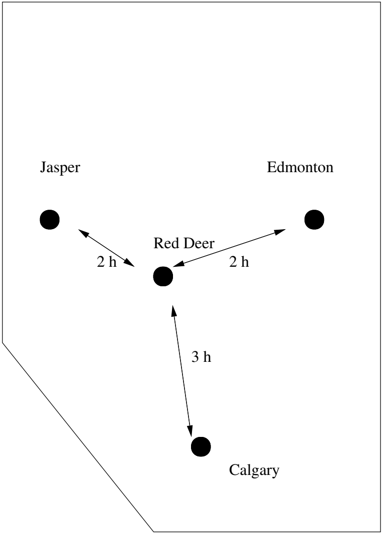

Timings Gaussian elimination for $\Delta = 4(10^{60} + 3)$

Naive Gauss			Dedicated strategy		
i	Col Nb	HNF time	i	Col Nb	HNF time
5	1067	357.9	5	1078	368.0
10	799	184.8	10	806	187.2
50	596	93.7	50	580	84.3
125	542	73.8	125	515	63.4
160	533	72.0	160	497	56.9
170	532	222.4	170	493	192.6

Regulator computation

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3. We augment the matrix M with the k extra rows

$$M' := \begin{pmatrix} M \\ \cdots\cdots\cdots \\ \vec{r_i} \end{pmatrix} \quad \vec{x_i}' := \left(\vec{x_i} \,\vdots\, 0 \ldots 0 \ -1 \ 0 \ldots 0 \right).$$

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The $\vec{x_i}'$ are kernel vectors of the new relation matrix M'.

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System solving in $O(L_\Delta(1/2, 3/\sqrt{8}))$.

Tab.: Regulator computation time for $\Delta = 4(10^n + 3)$

n	kernel computation	system solving
40	9.7	3.4
45	17.6	6.1
50	39.9	18.2
55	126.7	53.0
60	424.1	140.0
65	514.8	320.2
70	2728.5	791.1
75	8587.8	1775.8

Overall time comparison

Discriminants of the form $\Delta = 4(10^n + 3)$

TAB.: Overall time in seconds

n	Old	New
40	35.6	15.5
45	107.0	57.0
50	224.0	119.0
55	756.0	271.0
60	1535.0	655.0
65	24607.0	3125.0
70	38818.0	9991.0

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Conclusion

Thank you for your attention