Explicit Coleman integration for hyperelliptic curves

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Introduction: making sense of p-adic integrals

Let C be the hyperelliptic curve

$$y^2 = x^5 - x^4 + x^3 + x^2 - 2x + 1$$

over $\mathbf{Q}_7$ and let $P_1 = (0, 1), P_2 = (1, -1)$.

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1. How do we compute things like

$$\int_{P_1}^{P_2} \frac{dx}{2y}?$$

2. What do these (Coleman) integrals tell us?

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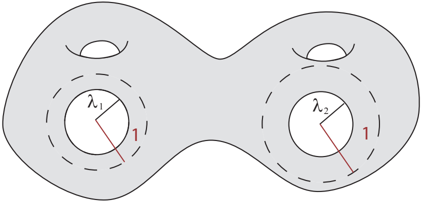
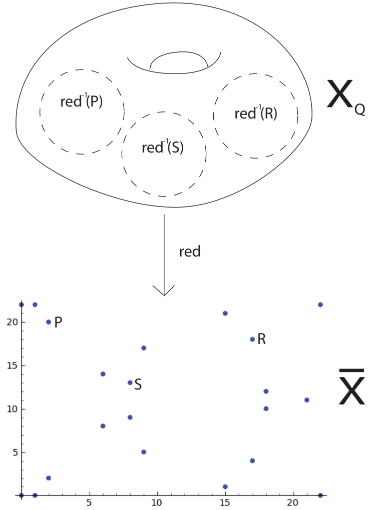
Notation and setup

- X: genus g hyperelliptic curve (of the form $y^2 = f(x)$ with $\deg f(x) = 2g+1$) over $K = \mathbf{Q}_p$
- p: prime of good reduction
- $\overline{X}$: special fibre of X
- $X_{\mathbf{Q}}$: generic fibre of X (as a rigid analytic space)

Notation and setup, in pictures

- There is a natural reduction map from $X_{\mathbf{Q}}$ to $\overline{X}$; the inverse image of any point of $\overline{X}$ is a subspace of $X_{\mathbf{Q}}$ isomorphic to an open unit disc. We call such a disc a *residue disc* of X.
- A *wide open subspace* of $X_{\mathbf{Q}}$ is the complement in $X_{\mathbf{Q}}$ of the union of a finite collection of disjoint closed discs of radius $\lambda_i < 1$:





Computing tiny integrals

We refer to any Coleman integral of the form $\int_P^Q \omega$ in which P, Q lie in the same residue disc as a *tiny integral*. To compute such an integral:

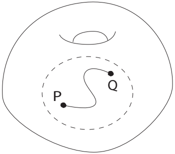
- Construct a linear interpolation from P to Q. For instance, in a non-Weierstrass residue disc, we may take

$$x(t) = (1-t)x(P) + tx(Q)$$
$$y(t) = \sqrt{f(x(t))},$$

where $y(t)$ is expanded as a formal power series in t.

- Formally integrate the power series in t:

$$\int_P^Q \omega = \int_0^1 \omega(x(t), y(t)).$$



Tiny integral: example

Let X be the hyperelliptic curve $y^2 = f(x) = x^5 - x^4 + x^3 + x^2 - 2x + 1$ over $\mathbf{Q}_7$, $\omega = \frac{dx}{2y}$, and

$$\begin{aligned} P &= (1, -1) \\ &= (1 + O(7^5), 6 + 6 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + 6 \cdot 7^4 + O(7^5)), \\ Q &= (1 + 7 + O(7^5), 6 + 4 \cdot 7 + 4 \cdot 7^2 + 3 \cdot 7^3 + 2 \cdot 7^4 + O(7^5)). \end{aligned}$$

We compute $\int_P^Q \omega$.

Tiny integral: example, continued

Computing $\int_P^Q \omega$:

1. Interpolate: we have

$$x(t) = (1-t)x(P) + tx(Q) = 1 + O(7^5) + \left(7 + O(7^5)\right) t$$

$$y(t) = \sqrt{f(x(t))} = 6 + 6\cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + 6 \cdot 7^4 + O(7^5) + \left(5 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + 6 \cdot 7^4 + O(7^5)\right) t + \cdots .$$

2. Integrate:

$$\int_P^Q \frac{dx}{2y} = \int_0^1 \frac{7 + O(7^5)}{(5 + 6\cdot 7 + \cdots) + (3 \cdot 7 + 6 \cdot 7^2 + \cdots) t + \cdots} dt = 3 \cdot 7 + 2 \cdot 7^3 + 5 \cdot 7^4 + O(7^5).$$

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Properties of the Coleman integral

Coleman formulated an integration theory on wide open subspaces of curves over $\mathcal{O}$, exhibiting no phenomena of path dependence.
This allows us to define $\int_P^Q \omega$ whenever ω is a meromorphic 1-form on X, and $P, Q \in X(\mathbf{Q}_p)$ are points where ω is holomorphic.
Properties of the Coleman integral include:

Theorem (Coleman)

- *Linearity:* $\int_P^Q (\alpha\omega_1 + \beta\omega_2) = \alpha \int_P^Q \omega_1 + \beta \int_P^Q \omega_2$.
- *Additivity:* $\int_P^R \omega = \int_P^Q \omega + \int_Q^R \omega$.
- *Change of variables: if X' is another such curve, and $f : U \to U'$ is a rigid analytic map between wide opens, then* $\int_P^Q f^*\omega = \int_{f(P)}^{f(Q)} \omega$.
- *Fundamental theorem of calculus:* $\int_P^Q df = f(Q) - f(P)$.

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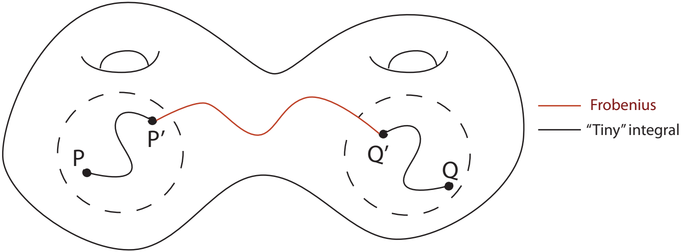
Coleman's construction

How do we integrate if P, Q aren't in the same residue disc?
Coleman's key idea: use Frobenius to move between different residue discs (Dwork's "analytic continuation along Frobenius")



So we need to calculate the action of Frobenius on differentials.

Frobenius, MW-cohomology

- X': affine curve ($X - \{$ Weierstrass points of $X\}$)
- A: coordinate ring of X'

To discuss the differentials we will be integrating, we recall:
The *Monsky-Washnitzer (MW) weak completion of* A is the ring $A^\dagger$ consisting of infinite sums of the form

$$\left\{ \sum_{i=-\infty}^{\infty} \frac{B_i(x)}{y^i},\ B_i(x) \in K[x], \deg B_i \leqslant 2g \right\},$$

further subject to the condition that $v_p(B_i(x))$ grows faster than a linear function of i as $i \to \pm\infty$. We make a ring out of these using the relation $y^2 = f(x)$.
These functions are holomorphic on wide opens, so we will integrate 1-forms

$$\omega = g(x,y)\frac{dx}{2y}, \quad g(x,y) \in A^\dagger.$$

Frobenius and a basis for de Rham cohomology

Any odd differential $\omega = g(x,y)\frac{dx}{2y}, g(x,y) \in A^\dagger$ can be written as

$$\omega = df + c_0\omega_0 + \cdots + c_{2g-1}\omega_{2g-1}, \tag{1}$$

where $f \in A^\dagger$, $c_i \in K$ and

$$\omega_i = \frac{x^i\,dx}{2y} \qquad (i = 0, \ldots, 2g-1). \tag{2}$$

That is, the ω_i form a basis of the odd part of the de Rham cohomology of $A^\dagger$. By linearity and the fundamental theorem of calculus, we reduce the integration of ω to the integration of the ω_i.

Integrals between points in non-Weierstrass discs

Let ϕ denote Frobenius. Recall that a *Teichmüller point* of $X_{\mathbf{Q}}$ is a point P such that $\phi(P) = P$.

One way to compute Coleman integrals $\int_P^Q \omega_i$:

- Find the Teichmüller points P', Q' in the residue discs of P, Q.
- Use Frobenius to compute $\int_{P'}^{Q'} \omega_i$.
- Use additivity in endpoints to recover the integral: $\int_P^Q \omega_i = \int_P^{P'} \omega_i + \int_{P'}^{Q'} \omega_i + \int_{Q'}^{Q} \omega_i$.

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Using Frobenius

More on Frobenius:

- Calculate the action of Frobenius ϕ on each basis differential, letting
$$\phi^*\omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j.$$
- Compute $\int_{P'}^{Q'} \omega_j$ by solving a linear system
- As the eigenvalues of the matrix M are algebraic integers of $\mathbf{C}_p$-norm $p^{1/2} \neq 1$, the matrix $M - I$ is invertible, and we may solve the system to obtain the integrals $\int_{P'}^{Q'} \omega_i$.

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Integrals via Teichmüller, continued

- The linear system gives us the integral between different residue discs.
- Putting it all together, we have

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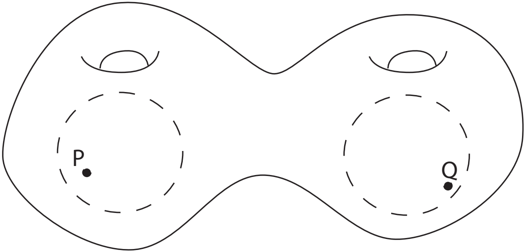

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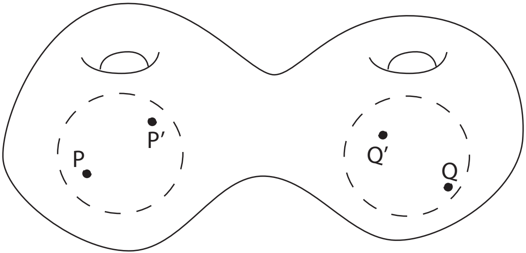

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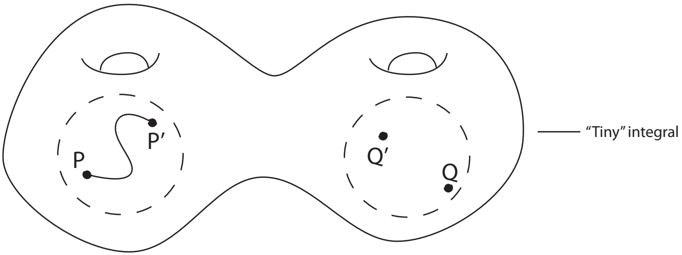

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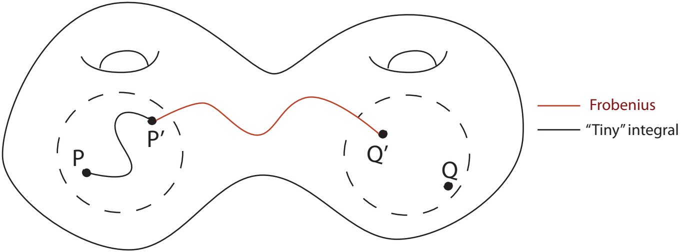

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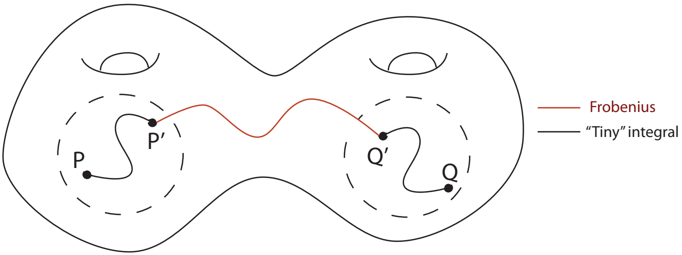

A different linear system

We could also bypass the computation of Teichmüller points by setting up the following linear system:

1. Calculate the action of Frobenius on each basis element:

$$(\phi^*)\omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j. \tag{3}$$

2. By change of variables, we obtain

$$\sum_{j=0}^{2g-1} (M-I)_{ij} \int_P^Q \omega_j = f_i(P) - f_i(Q) - \int_P^{\phi(P)} \omega_i - \int_{\phi(Q)}^{Q} \omega_i. \tag{4}$$

3. Solving the linear system yields $\int_P^Q \omega_j = (M-I)^{-1}\left(f_i(P) - f_i(Q) - \int_P^{\phi(P)} \omega_i - \int_{\phi(Q)}^{Q} \omega_i\right).$

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Weierstrass endpoints of integration

Suppose now that P, Q lie in different residue discs, at least one of which is Weierstrass.

Proposition

Let ω be an odd, everywhere meromorphic differential on X. Choose $P, Q \in X(\mathbf{C}_p)$ which are not poles of ω, with P Weierstrass. Then for ι the hyperelliptic involution, $\int_P^Q \omega = \frac{1}{2}\int_{\iota(Q)}^Q \omega$. In particular, if Q is also a Weierstrass point, then $\int_P^Q \omega = 0$.

Numerical examples: torsion points (Leprévost)

Leprévost showed that the divisor $(1, -1) - \infty^+$ on the genus 2 curve $y^2 = (2x-1)(2x^5 - x^4 - 4x^2 + 8x - 4)$ over $\mathbf{Q}$ is torsion of order 29. The integrals of holomorphic differentials against this divisor must vanish. Indeed, let

$$C : y^2 = x^5 + \frac{33}{16}x^4 + \frac{3}{4}x^3 + \frac{3}{8}x^2 - \frac{1}{4}x + \frac{1}{16}$$

be the pullback of Leprévost's curve by the linear fractional transformation $x \mapsto (1-2x)/(2x)$ taking ∞ to $1/2$. The original points $(1, -1), \infty^+$ correspond to the points $P = (-1, 1)$, $Q = (0, \frac{1}{4})$ on C. The curve C has good reduction at $p = 11$, and we compute

$$\int_P^Q \omega_0 = \int_P^Q \omega_1 = O(11^6), \int_P^Q \omega_2 = 7\cdot 11 + 6\cdot 11^2 + 3\cdot 11^3 + 11^4 + 5\cdot 11^5 + O(11^6),$$

consistent with the fact that $Q - P$ is torsion and ω_0, ω_1 are holomorphic but ω_2 is not.

Numerical examples: Chabauty method

We give an example arising from the Chabauty method, taken from "The method of Chabauty and Coleman" (McCallum-Poonen).
Let X be the curve

$$y^2 = x(x-1)(x-2)(x-5)(x-6),$$

whose Jacobian has Mordell-Weil rank 1. The curve X has good reduction at 7, and

$$X(\mathbf{F}_7) = \{(0,0),(1,0),(2,0),(5,0),(6,0),(3,6),(3,-6),\infty\}.$$

By Theorem 5.3(2) of [McC-P], we know $|X(\mathbf{Q})| \leqslant 10$. However, we can find 10 rational points on X: the six rational Weierstrass points, and the points $(3,\pm 6),(10,\pm 120)$. Hence $|X(\mathbf{Q})| = 10$.

Chabauty method, continued

Since the Chabauty condition holds, there must exist a holomorphic differential ω for which $\int_\infty^Q \omega = 0$ for all $Q \in X(\mathbf{Q})$. We can find such a differential by taking Q to be one of the rational non-Weierstrass points, then computing $a := \int_\infty^Q \omega_0, b := \int_\infty^Q \omega_1$ and setting $\omega = b\omega_0 - a\omega_1$. For $Q = (3,6)$, we obtain

$$a = 6 \cdot 7 + 6 \cdot 7^2 + 3 \cdot 7^3 + 3 \cdot 7^4 + 2 \cdot 7^5 + O(7^6)$$
$$b = 4 \cdot 7 + 2 \cdot 7^2 + 6 \cdot 7^3 + 4 \cdot 7^5 + O(7^6).$$

We then verify that $\int_Q^R \omega = 0$ for each of the other rational points R.

Future directions

- Iterated integrals
  - Can define
$$\int_P^Q \omega_n \cdots \omega_1 = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_n) \cdots f_1(t_1)\, dt_n \cdots dt_1$$
    which appear in applications of Coleman integration, e.g., p-adic regulators in K-theory, and the nonabelian Chabauty method
- Beyond hyperelliptic curves
  - Convert algorithms for computing Frobenius actions on de Rham cohomology (Gaudry-Gürel, Castryck-Denef-Vercauteren) into algorithms for computing Coleman integrals on such curves
- Heights after Harvey
  - Our algorithms have linear runtime dependence on the prime p, arising from the corresponding dependence in Kedlaya's algorithm; could possibly follow Harvey's variant of Kedlaya's algorithm to reduce this to square-root dependence on p

Applications of explicit Coleman integration

- p-adic heights on curves: $h_p(D_1, D_2) = \int_{D_2} \omega_{D_1}$
- Syntomic regulators on curves: for $\{f, g\} \in K_2(C)$, $\mathrm{reg}_p(\{f, g\})(\omega) = \int_{(f)} \log(g)\omega$
- p-adic polylogarithms and multiple zeta values, following Besser-de Jeu
- Experiments with Chabauty's method: find P such that $\int_0^P \omega = 0$
- Torsion points on curves (Coleman's original application, for curves of $g > 1$)
- Kim's nonabelian Chabauty method: use $\int_b^z \omega_0\omega_1$ to recover integral points on elliptic curves